

MATROIDS OVER A RING

ALEX FINK AND LUCA MOCI

ABSTRACT. We introduce the notion of a matroid M over a commutative ring R , assigning to every subset of the ground set a finitely generated R -module according to some axioms. When R is a field, we recover matroids. When $R = \mathbb{Z}$, and when R is a DVR, we get (structures which contain all the data of) quasi-arithmetic matroids, and valuated matroids, respectively.

More generally, whenever R is a Dedekind domain, we extend all the usual properties and operations holding for matroids (e.g., duality), and we compute the Tutte-Grothendieck group of matroids over R .

1. INTRODUCTION

The notion of a *matroid* axiomatizes the linear algebra of a list of vectors. Matroid theory has proved to be a versatile language to deal with many problems on the interfaces of combinatorics and algebra. In the years since 1935, when Whitney first introduced matroids, a number of enriched variants thereof have arisen, among them oriented matroids [2], valuated matroids [9], complex matroids [1], and (quasi-)arithmetic matroids [16, 6]. Each of these structures retains some information about a vector configuration, or an equivalent object, which is richer than the purely linear algebraic information that matroids retain.

As a running motivating example, let us focus on quasi-arithmetic matroids. A quasi-arithmetic matroid endows a matroid with a multiplicity function, whose values are the cardinalities of certain finite abelian groups, namely, the torsion parts of the quotients of an ambient lattice \mathbb{Z}^n by the sublattices spanned by subsets of vectors. From a list of vectors with integer coordinates one may produce objects like a toric arrangement, a partition function, and a zonotope (see [8]). In order to have a combinatorial structure from which these objects may be read off, one needs to keep track of arithmetic properties of the vectors, and this is what quasi-arithmetic matroids provide. (For the difference between quasi-arithmetic and arithmetic matroids, see Remark 6.4.)

It is natural to ask to what extent these generalizations of matroids can be unified under a common framework. Such a unification was sought by Dress in his program of matroids with coefficients, represented for example in his work with Wenzel [9] wherein valuated matroids are matroids with coefficients in a “fuzzy ring”.

In the present paper we suggest a different approach to such unification, by defining the notion of a *matroid M over a commutative ring R* . Such an M assigns, to every subset A of a ground set, a finitely generated R -module $M(A)$ according to some axioms (Definition 2.1). We find this definition to have multiple agreeable features. For one, by building on the well-studied setting of modules over commutative rings, we get a theory where the considerable power and development of commutative algebra can be easily brought to bear. For another, unlike arithmetic

and valuated matroids, a matroid over R is not defined as a matroid decorated with extra data; there are only two axioms, and we suggest that they are comparably simple to the matroid axioms themselves. Indeed, a *representable* matroid over R is precisely a vector configuration in a finitely generated R -module, and the axioms of a matroid over R say only that minors of at most two elements are such representable matroids — that is, matroids are *locally* representable matroids, the locality interpreted in the Boolean lattice of sets A .

When R is a field, a matroid M over R is nothing but a matroid: the data $M(A)$ is a vector space, which contains only the information of its dimension, and this directly encodes the rank function of M . When $R = \mathbb{Z}$, every module $M(A)$ is an abelian group, and by extracting its torsion subgroup we get a quasi-arithmetic matroid. When R is a discrete valuation ring (DVR), we may similarly extract a valuated matroid. More generally, whenever R is a Dedekind domain, we can extend all the usual properties and operations holding for matroids, such as duality.

The idea of matroids over rings was suggested by certain features of the theory of quasi-arithmetic matroids. Some significant information about an integer vector configuration is lost in passing to the multiplicity function, as there exist many finite abelian groups with the same cardinality. Recording the whole structure of these groups is more desirable in several situations, for example, in developing a combinatorial intersection theory for the arrangements of subtori arising as characteristic varieties. The properties of the multiplicity function of a quasi-arithmetic matroid turn out to be just shadows of group-theoretic properties.

One of the most-loved invariants of matroids is their Tutte polynomial $\mathbf{T}_M(x, y)$. It thus comes as no surprise that the Tutte polynomial has been considered for generalizations of matroids as well. A quasi-arithmetic matroid \hat{M} has an associated arithmetic Tutte polynomial $\mathbf{M}_{\hat{M}}(x, y)$, which has proved to be a useful tool in studying toric arrangements, partition functions, zonotopes, and graphs ([16, 7, 3]). More strongly, the authors of [3] define a *Tutte quasi-polynomial* of an integer vector configuration, interpolating between $\mathbf{T}_M(x, y)$ and $\mathbf{M}_{\hat{M}}(x, y)$, which is no longer an invariant of the quasi-arithmetic matroid (as it depends on the groups, not just their cardinalities).

Among its properties, the Tutte polynomial of a classical matroid is the universal deletion-contraction invariant. In more algebraic language, following [4], the class of a matroid in the Tutte-Grothendieck group for deletion-contraction relations is exactly its Tutte polynomial. While the arithmetic Tutte polynomial and Tutte quasi-polynomial are deletion-contraction invariants, neither is universal for this property. Our generalization of the Tutte polynomial for matroids over a Dedekind ring R is also the class in the Tutte-Grothendieck group, so it retains the universality of the usual Tutte polynomial, and we obtain the two generalizations of Tutte just mentioned as evaluations of it.

This paper is organized as follows. In Section 2 we give the basic definitions for matroids and polymatroids over a commutative ring, including representability, and we explain how they generalize the classical ones.

We introduce the assumption that R is a Dedekind domain, and do some groundwork, in Section 3. This assumption on R remains for the most part in force from this section onward. Its first application comes in Section 4, where we establish

the existence (Definition 4.1, Proposition 4.4) and the properties of the dual of a matroid over a Dedekind domain R .

In Section 5 we develop the local theory, by proving a structure theorem for matroids over a DVR (Propositions 5.2 and 5.4). We show connections with the Hall algebra and with the tropical Plücker relations for the flag variety. Finally, we describe how to recover valuated matroids.

The global theory is developed in Section 6 (Propositions 6.1 and 6.2). This also explains the connection between matroids over \mathbb{Z} and quasi-arithmetic matroids (Corollary 6.3).

Finally, in Section 7 we compute the Tutte-Grothendieck group (Theorem 7.1). In particular, given a matroid over \mathbb{Z} , we present its Tutte quasi-polynomial as an evaluation of its class in $K(\mathbb{Z}\text{-Mat})$.

Acknowledgments. The authors thank Ezra Miller for helpful conversations.

2. MATROIDS OVER A RING

By $R\text{-Mod}$ we mean the category of finitely generated R -modules over a commutative ring R . We will feel free to write “f.g.” for “finitely generated” throughout.

Definition 2.1. Let R be a commutative ring. A *matroid over R* on the ground set E is a function M assigning to each subset $A \subseteq E$ a finitely-generated R -module $M(A)$ satisfying the following axioms:

- (M1) For any $A \subseteq E$ and $b \in E \setminus A$, there exists a surjection $M(A) \twoheadrightarrow M(A \cup \{b\})$ whose kernel is a cyclic submodule of $M(A)$.
- (M2) For any $A \subseteq E$ and $b \neq c \in E \setminus A$, there exists a pushout

$$\begin{array}{ccc} M(A) & \longrightarrow & M(A \cup \{b\}) \\ \downarrow & \lrcorner & \downarrow \\ M(A \cup \{c\}) & \longrightarrow & M(A \cup \{b, c\}) \end{array}$$

where all four morphisms are surjections with cyclic kernel.

It is clear that axiom (M2) implies axiom (M1) when the ground set has at least two elements.

Definition 2.2. A *polymatroid over R* on the ground set E is a function M assigning to each subset $A \subseteq E$ a finitely-generated R -module $M(A)$ satisfying the following axiom:

- (PM) For any $A \subseteq E$ and $b \neq c \in E \setminus A$, there exists a pushout

$$\begin{array}{ccc} M(A) & \longrightarrow & M(A \cup \{b\}) \\ \downarrow & \lrcorner & \downarrow \\ M(A \cup \{c\}) & \longrightarrow & M(A \cup \{b, c\}) \end{array}$$

where all four maps are surjections.

Clearly, the choice of the modules $M(A)$ is only relevant up to isomorphism. We regard (poly)matroids M and M' over R to be equal if they are on the same ground set E and $M(A) \cong M'(A)$ for all $A \subseteq E$.

For notational concision, we will hereafter let $M(Ab)$ abbreviate $M(A \cup \{b\})$, $M(Abc)$ stand for $M(A \cup \{b, c\})$, and so forth.

Remark 2.3. The fact that

$$\begin{array}{ccc} M(A) & \xrightarrow{\varphi} & M(Ab) \\ \psi \downarrow & & \downarrow \psi' \\ M(Ac) & \xrightarrow{\varphi'} & M(Abc) \end{array}$$

is a pushout diagram of modules can be restated as

$$M(Abc) \simeq \frac{M(Ac) \oplus M(Ab)}{\{(\psi(x), -\varphi(x)), x \in M(A)\}}.$$

The fact that the maps are surjections implies

$$M(Abc) \simeq M(A)/(\ker \varphi, \ker \psi).$$

where by $(\ker \varphi, \ker \psi)$ we denote the submodule of $M(A)$ generated by the two kernels. \diamond

Example 2.4. Not every polymatroid over R satisfying (M1) is a matroid over R . For a counterexample, let $R = \mathbb{Z}$. There is a pushout diagram of surjections

$$(2.1) \quad \begin{array}{ccc} \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \longrightarrow & \mathbb{Z}/2\mathbb{Z} \\ \downarrow & \lrcorner & \downarrow \\ \mathbb{Z}/4\mathbb{Z} & \longrightarrow & \mathbb{Z}/2\mathbb{Z} \end{array}$$

in which the top map has kernel $\langle (2, 0), (0, 1) \rangle$ and the left map has kernel $\langle (2, 1) \rangle$. Moreover there exist surjections $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \twoheadrightarrow \mathbb{Z}/2\mathbb{Z}$ with cyclic kernel: there are two such, one with kernel $\langle (1, 0) \rangle$ and one with kernel $\langle (1, 1) \rangle$. However, neither of these maps can be fitted into a pushout diagram of surjections with groups isomorphic to (2.1); both those pushouts are the trivial group. So diagram (2.1) corresponds to a function from $\mathcal{B}(2)$ to $\mathbb{Z}\text{-Mod}$ that satisfies (PM) and (M1) but not (M2). \diamond

The fundamental way of producing matroids over R is from vector configurations in an R -module. Given a f.g. R -module N and a list $X = x_1, \dots, x_n$ of elements of N , the matroid M_X of X associates to the sublist A of X the quotient

$$(2.2) \quad M_X(A) = N / \left(\sum_{x \in A} Rx \right).$$

For each $x \in X$ there is a quotient map from $M_X(A)$ to $M_X(A \cup \{x\})$, which quotients out by the image of Rx in $M_X(A)$. This single system of maps satisfies axioms (M1) and (M2); the choice of the map $M(A) \rightarrow M(Ab)$ in (M2) is independent of c , and similarly for the other maps.

The following definition captures this concisely. Let $\mathcal{B}(E)$ be the category of the Boolean poset of subsets of E , where inclusions of sets are the morphisms.

Definition 2.5. A matroid M over R is *representable* (or *realizable*) if it is the map on objects of some functor $F : \mathcal{B}(E) \rightarrow R\text{-Mod}$, and axioms (M1) and (M2) are satisfied by choosing the morphisms $F(A \rightarrow Ab)$. A *representation* (*realization*) of M is a choice of such an F .

So M_X is a representable matroid, and X gives a representation thereof. We have chosen to cast Definition 2.5 as we did, as opposed to in a more down-to-earth way involving M_X , to emphasize the way in which a representable matroid is a matroid. A representation of a matroid over R is a functor from $\mathcal{B}(E)$, with both objects and morphisms having images. A general matroid over R is what is gotten by retaining only the objects as data, discarding the morphisms and merely requiring that they can be resupplied to look like a represented matroid over R in any square of covering relations in $\mathcal{B}(E)$.

Fact 2.6. If a matroid M over R is representable, corresponding to the functor F , then it is the matroid M_X of a vector configuration $(N, X = \{x_a\})$, where N is $F(\emptyset)$, and x_a is a generator of $\ker F(\emptyset \rightarrow \{a\})$ for each $a \in E$. Indeed, in this above setting, the pushout axiom (M2) applied to F guarantees that equation (2.2) holds for all $A \subseteq E$.

Our having chosen to call these objects “matroids over R ” is appropriate, as they are a generalization of matroids in the classical sense, as we show in Proposition 2.9. There is one hitch in the equivalence, corresponding to the ability to choose a vector configuration that does not span its ambient space. Accordingly, let us say that a matroid M over R is *full-rank* if no nontrivial projective module is a direct summand of $M(E)$. Lemma 2.8 shows that very little is lost in restricting to full-rank matroids.

Before getting there we must generalize some standard operations on matroids. In several cases this is straightforward, but duality is conspicuously not among these: for matroid duality to work well, we must assume that R is a Dedekind domain, and so we treat it in Section 4.

Let M and M' be matroids over R on respective ground sets E and E' . We define their *direct sum* $M \oplus M'$ on the ground set $E \amalg E'$ by

$$(M \oplus M')(A \amalg A') = M(A) \oplus M'(A').$$

If i is an element of E , we define two matroids over R on the ground set $E \setminus \{i\}$: the *deletion* of i in M , denoted $M \setminus i$, by

$$(M \setminus i)(A) = M(A)$$

and the *contraction* of i in M , denoted M / i , by

$$(M / i)(A) = M(A \cup \{i\}).$$

It is easy to check that these satisfy axioms (M1) and (M2); they are entirely inherited except for the cases of (M2) for $M \oplus M'$ with one of b and c in E and the other in E' , but these cases are clear. Since these constructions can be made without reliance on the axioms (M1) and (M2), we will sometimes use them in the sequel when speaking of a map $\mathcal{B}(E) \rightarrow R\text{-Mod}$ which has not yet been shown to be a matroid over R .

The next fact is immediate from these definitions.

Fact 2.7. The class of representable matroids is closed under minors and direct sums:

- (a) If M is represented by the functor $F : \mathcal{B}(E) \rightarrow R\text{-Mod}$, then $M/A \setminus B$ is represented by the restriction of F to the subposet of sets containing A and disjoint from B .
- (b) If M_i is represented by the vector configuration X_i within a module N_i , for $i = 1, 2$, then $M_1 \oplus M_2$ is represented by the configuration $(X_1, 0) \cup (0, X_2)$ within $N_1 \oplus N_2$.

If N is an R -module, let the *empty matroid* for N be the matroid over R on the ground set \emptyset which maps \emptyset to N . By a *projective empty matroid* we mean an empty matroid for a projective module.

Lemma 2.8. *Every matroid M over R is the direct sum of a full-rank matroid over R and a projective empty matroid.*

Note that this decomposition will be unique in the Dedekind setting, as a consequence of Proposition 3.3.

Proof. Suppose M is not full-rank, so that some projective module P is a direct summand of $M(E)$. Then in fact P is a direct summand of every module $M(A)$, since this property lifts back along the surjections in axiom (M1). Therefore M is a direct sum of another matroid M' over R and the empty matroid for P . Since M is finitely generated, iterating this process with M' in place of M eventually reaches a full-rank matroid. \square

Recall that the *corank* $\text{cork}(A)$ of a set A in a classical matroid is equal to $\text{rk}(E) - \text{rk}(A)$, where $\text{rk}(E)$ is the rank of the matroid.

Proposition 2.9. *Let \mathbb{K} be a field. Full-rank matroids M over \mathbb{K} are equivalent to (classical) matroids. If M is a full-rank matroid over \mathbb{K} , then $\dim M(A)$ is the corank of A in the corresponding classical matroid.*

A matroid over \mathbb{K} is representable if and only if, as a classical matroid, it is representable over \mathbb{K} .

Proof. The finitely generated modules over \mathbb{K} are the finite-dimensional \mathbb{K} -vector spaces, which are completely classified up to isomorphism by dimension. So we may replace $M(A)$ by its \mathbb{K} -dimension without losing information.

We now check that the conditions on the dimensions of the $M(A)$ given by axioms (M1) and (M2) and the full-rank condition are equivalent to the following set of rank axioms for matroids, recast in terms of a corank function $\text{cork} : 2^E \rightarrow \mathbb{N}$:

- (C0) $\text{cork}(E) = 0$.
- (C1) For $A \subseteq E$ and $b \in E \setminus A$, $\text{cork}(A) - \text{cork}(Ab)$ equals 0 or 1.
- (C2) For $A \subseteq E$ and $b \neq c \in E \setminus A$,

$$\text{cork}(A) + \text{cork}(Abc) \geq \text{cork}(Ab) + \text{cork}(Ac).$$

Axiom (C0) is the full-rank condition. For x in a \mathbb{K} -vector space V , the difference $\dim V - \dim(V/\langle x \rangle)$ equals zero if x is zero and one otherwise, so that (M1) and (C1) are equivalent.

Finally, if maps among the vector spaces are chosen to give the square of axiom (M2), let K and L be the respective kernels of $M(A) \rightarrow M(Ab)$ and $M(A) \rightarrow$

$M(Ac)$. Then $M(Abc) = M(A)/(K \cup L)$. By arranging K and L suitably, their union $K \cup L$ can be chosen to have any dimension from $\min(\dim K, \dim L)$ to $\dim K + \dim L$ inclusive (except those that exceed $\dim M(A)$), but no others. That is, the only conditions on $\dim M(Abc)$ in terms of the other dimensions are the monotonicity conditions $\dim M(Abc) \leq \min \dim M(Ab)$, and the submodularity condition

$$\dim M(A) + \dim M(Abc) \geq \dim M(Ab) + \dim M(Ac),$$

which is (C2). Since (C1) implies monotonicity, we see that (M2) and (C2) are equivalent.

The representability claim is already proved by our prior observation that a represented matroid over \mathbb{K} embodies a \mathbb{K} -vector configuration. \square

We record also the analogue for polymatroids. The corank function of a polymatroid is axiomatized like that for a matroid, replacing (C1) with

$$(PC) \text{ For } A \subseteq E \text{ and } b \in E \setminus A, \text{cork}(A) \geq \text{cork}(Ab).$$

Proposition 2.10. *Let \mathbb{K} be a field. Polymatroids M over \mathbb{K} with $M(E) = 0$ are equivalent to (classical) polymatroids. In this case $\dim M(A)$ is the corank of A in the corresponding polymatroid.*

Proof. The previous proof goes through *mutatis mutandis*, by considering axioms (PC) and (PM). \square

Let $R \rightarrow S$ be a map of rings. Then every matroid over S is naturally also a matroid over R . Furthermore, given such a map $R \rightarrow S$, the tensor product $— \otimes_R S$ is a functor $R\text{-Mod} \rightarrow S\text{-Mod}$. One can use this to perform base change of matroids over R . If M is a matroid over R , define $M \otimes_R S$ be the composition of M with $— \otimes_R S$, so that

$$(M \otimes_R S)(A) = M(A) \otimes_R S$$

for all A . (As with other uses of the tensor product, we will omit the subscript R in the notation where this causes no unclarity.)

Proposition 2.11. *If M is a matroid over R , then $M \otimes_R S$ is a matroid over S .*

Proof. Let $0 \rightarrow K \rightarrow N \rightarrow N' \rightarrow 0$ be a short exact sequence of R -modules, with K cyclic. Tensor product being right exact, we get an exact sequence $K \otimes S \rightarrow N \otimes S \rightarrow N' \otimes S \rightarrow 0$, so the kernel of $N \otimes S \rightarrow N' \otimes S$ is a quotient of the cyclic S -module $K \otimes S$, and is therefore cyclic. Thus axiom (M1) for M implies axiom (M1) for $M \otimes_R S$.

Since tensor product is a left adjoint functor (to Hom), it preserves pushouts. Thus axiom (M2) for M implies axiom (M2) for $M \otimes_R S$. \square

As one immediate application, if $R \rightarrow \mathbb{K}$ is a map to a field, one can construct from a matroid M over R a classical matroid $M \otimes_R \mathbb{K}$ (except that the result may not be full-rank). For example, if $\text{Frac}(R)$ denotes the fraction field of R , then $M \otimes \text{Frac}(R)$ is a classical matroid; we will call it the *generic matroid* of M . Another instance is when \mathbb{K} is the residue field of any prime of R .

Question 2.12. There are various ways to axiomatize matroids using rank functions. It is possible to state the submodularity axiom for matroids as our axiom (C2) parallelling (M2), that is

$$\text{rank}(Ab) + \text{rank}(Ac) \geq \text{rank}(A) + \text{rank}(Abc),$$

but the more usual statement of this axiom doesn't restrict to covers in the Boolean lattice: it asserts that

$$\text{rank}(A) + \text{rank}(B) \geq \text{rank}(A \cap B) + \text{rank}(A \cup B)$$

for all $A, B \subseteq E$. Similarly, the fact that rank is nondecreasing and bounded by cardinality can be framed on covers, like our (M1), or on all containments. *Is there an axiomatization of matroids over R which replaces (M1) and (M2) with axioms on all containments, respectively pairs of sets?*

Example 2.4 suggests that such an axiom system would still need to make reference to the number of generators of kernels. It is also conceivable that the axioms sought would only agree over Dedekind domains: the behaviour exhibited in Example 3.2 below for a non-Dedekind domain interferes with naïve attempts to patch pushout squares together. \diamond

Remark 2.13. It appears that our construction of matroids over rings can be patched in a sheaf-theoretic fashion to yield sheaves of matroids over schemes. Proper investigation of these sheaves of matroids is left to future work. \diamond

3. DEDEKIND DOMAINS

In several ways, Definition 2.1 yields a theory best parallelling the theory of classical matroids just when R is a Dedekind domain. In this section we review some properties of these rings for use in following sections.

One well-behaved feature of Dedekind domains in our setting is framed in Lemma 3.1: morally, over a Dedekind domain R , relations among isomorphism types of modules (like “is a quotient by one element of”) impose strong enough restrictions on the combinatorics for our theory of matroids over R to work. Example 3.2 shows that this fails in the two-dimensional setting.

Lemma 3.1. *Let R be a Dedekind domain. Given two R -modules N and N' , all cyclic modules that appear as kernels of surjections $N \twoheadrightarrow N'$ are isomorphic.*

Proof of Lemma 3.1. Suppose we have two surjections $N \twoheadrightarrow N'$ with kernels respectively generated by elements x and y of N . We show that $\langle x \rangle$ and $\langle y \rangle$ are isomorphic as R -modules with the isomorphism given by $x \mapsto y$. It is enough to show that this map is an isomorphism after localizing at every maximal prime \mathfrak{m} of R . Now, the isomorphism class of $\langle x \rangle_{\mathfrak{m}}$ can be read off of the other two modules in the localized exact sequence

$$0 \rightarrow \langle x \rangle_{\mathfrak{m}} \rightarrow N_{\mathfrak{m}} \rightarrow N'_{\mathfrak{m}} \rightarrow 0.$$

To be precise, if the rank of $N_{\mathfrak{m}}$ exceeds that of $N'_{\mathfrak{m}}$, then $\langle x \rangle_{\mathfrak{m}} \cong R_{\mathfrak{m}}$ is free; otherwise, $\langle x \rangle_{\mathfrak{m}}$ is torsion and is determined up to isomorphism by its (R/\mathfrak{m}) -dimension, which is the difference of the dimensions of the torsion parts of $N_{\mathfrak{m}}$ and $N'_{\mathfrak{m}}$. The isomorphism class of $\langle y \rangle_{\mathfrak{m}}$ is determined in the same way from the same data, so that $\langle x \rangle_{\mathfrak{m}} \cong \langle y \rangle_{\mathfrak{m}}$ for all \mathfrak{m} . And since x and y are generators, the isomorphism can be taken to send $x \mapsto y$. \square

Example 3.2. Let $R = \mathbb{K}[x, y]/\langle x, y \rangle^2$, the ring of two-dimensional first-order jets, which is imprecisely the “smallest” two-dimensional ring. Let N be the length 3 R -module $\langle x, y \rangle / \langle x^2, y^2 \rangle$, where these x and y should be read as elements not of R but of $\mathbb{K}[x, y]$ (thus N is isomorphic to the so-called Matlis dual of R). Then the quotients $N/\langle x \rangle$ and $N/\langle y \rangle$ are both isomorphic to \mathbb{K} , but their kernels $\langle x \rangle / \langle x^2, xy^2 \rangle$ and $\langle y \rangle / \langle x^2y, y^2 \rangle$ are not isomorphic. \diamond

We next recall some structural results about R -modules. Given an R -module N , let $N_{\text{tors}} \subseteq N$ denote the submodule of its torsion elements, and N_{proj} denote the projective module N/N_{tors} , so that $N \cong N_{\text{tors}} \oplus N_{\text{proj}}$. When we speak of the rank of N , we mean the rank of N_{proj} , i.e. the dimension of $N \otimes \text{Frac}(R)$.

Proposition 3.3. [10, exercises 19.4–6] *Every nonzero f.g. projective module over a Dedekind domain R is uniquely isomorphic to $R^k \oplus I$ for some $k \geq 0$ and nonzero ideal I , up to differing isomorphic choices of I . In particular, for ideals I and J , we have $I \oplus J \cong R \oplus (I \otimes J)$.*

Every f.g. R -module N is the direct sum of a projective module isomorphic to N_{proj} , and its torsion submodule N_{tors} . This torsion module may be written uniquely up to isomorphism as a sum of submodules R/\mathfrak{m}^k for \mathfrak{m} a maximal prime of R and $k \in \mathbb{Z}_{>0}$. It also may be written uniquely as a sum of submodules $R/I_1 \oplus \cdots \oplus R/I_m$ (its invariant factors) for a chain $I_1 \subseteq \cdots \subseteq I_m$ of ideals of R .

We will also find useful a description of the algebraic K -theory group $K_0(R)$ of f.g. R -modules: that is, the abelian group generated by classes $[N]$ of f.g. R -modules, modulo the relations $[N] = [N'] + [N'']$ for any exact sequence

$$0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0.$$

Proposition 3.4. *The group $K_0(R)$ is isomorphic to $\mathbb{Z} \oplus \text{Pic}(R)$.*

Proof. In Corollary 2.6.3 of [20] the K -theory group $K^0(R)$ of *projective* R -modules is shown to be $\mathbb{Z} \oplus \text{Pic}(R)$, via the map $[P] = (\text{rk}(P), \det(P))$. But since R is a regular ring, the natural homomorphism $K^0(R) \rightarrow K_0(R)$ is an isomorphism [11, §15.1]. \square

We will speak mostly of the second summand of $K_0(R)$, and so for a f.g. R -module N we will write $[N]$ for its image in the summand $\text{Pic}(R)$ of $K_0(R)$. Note in particular that this extends the usual map from invertible ideals to $\text{Pic}(R)$. The potential nontriviality of this summand $\text{Pic}(R) \subseteq K_0(R)$ has global consequences for matroids over R : see Proposition 4.7 below.

4. DUALITY FOR MATROIDS OVER DEDEKIND DOMAINS

In this section R will be a Dedekind domain. Let M be a matroid over R , on ground set E . For any $A \subseteq E$ and $b \in E \setminus A$, the map provided by axiom (M1) may be fitted into an exact sequence of the shape

$$(4.1) \quad 0 \rightarrow I \rightarrow R \rightarrow M(A) \rightarrow M(Ab) \rightarrow 0$$

where R/I is chosen isomorphic to the cyclic kernel from axiom (M1). Here I is a projective R -module, possibly zero but of rank at most 1. Also, since R has global

dimension 1, a minimal projective resolution of $M(\emptyset)$ has length at most 1. Label its terms

$$(4.2) \quad 0 \rightarrow P_1^\emptyset \rightarrow P_0^\emptyset \rightarrow M(\emptyset) \rightarrow 0.$$

From any maximal flag of subsets $\emptyset = A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_{|A|} = A$ we obtain a composite map

$$P_0^\emptyset \rightarrow M(\emptyset) \rightarrow M(A_1) \rightarrow \cdots \rightarrow M(A).$$

The kernel of this composition $P_0^\emptyset \rightarrow M(A)$ has a filtration whose subquotients are the kernels of the individual arrows in it. This allows us to resolve $\ker P_0^\emptyset \rightarrow M(A)$ with a correspondingly filtered resolution:

$$P(A)_\bullet: \quad 0 \rightarrow P_2 \rightarrow P_1^\emptyset \oplus R^{|A|} \xrightarrow{d_1} P_0^\emptyset \rightarrow M(A) \rightarrow 0$$

The subquotient complexes appearing in the filtration of the part $0 \rightarrow P_2 \rightarrow P_1^\emptyset \oplus R^{|A|}$ of this complex are one copy of $0 \rightarrow 0 \rightarrow P_1^\emptyset$, from (4.2), and $|A|$ copies of complexes $0 \rightarrow I \rightarrow R$, from (4.1). In particular P_2 is a projective module of rank $|A| - \text{rk}(A)$, where $\text{rk}(A)$ is the rank of A in the generic matroid $M \otimes \text{Frac}(R)$, and the rank $|A| - \text{rk}(A)$ counts those surjections $A_i \rightarrow A_{i+1}$ whose kernel R/I is a proper quotient of R .

The complex $P(A)_\bullet$ is also a projective resolution of $M(A)$. For convenience, when viewed in this role, we will also write it

$$P(A)_\bullet: \quad 0 \rightarrow P_2(A) \rightarrow P_1(A) \xrightarrow{d_1} P_0(A) \rightarrow M(A) \rightarrow 0.$$

As usual, we write $^\vee$ for the contravariant functor $\text{Hom}(_, R)$.

Definition 4.1. Define the module $M^*(E \setminus A)$ as the cokernel of the map dual to d_1 in $P(A)_\bullet$, that is

$$M^*(E \setminus A) \doteq \text{coker} (P_0(A)^\vee \xrightarrow{d_1^\vee} P_1(A)^\vee).$$

We define M^* , the *dual* matroid over R to M , to be the collection of these modules $M^*(E \setminus A)$.

Lemma 4.2. *The module $M^*(E \setminus A)$ is well-defined.*

We begin by noting

Lemma 4.3. *For any exact sequence*

$$0 \rightarrow K_2 \rightarrow Q_1 \rightarrow Q_0 \rightarrow N \rightarrow 0$$

of R -modules with Q_1 and Q_0 projective, the cokernel of the induced map $Q_0^\vee \rightarrow Q_1^\vee$ is isomorphic to $\text{Ext}^1(N, R) \oplus \text{Hom}(K_2, R)$.

Proof. Let K_1 be the kernel of $Q_0 \rightarrow N$. This splits the given sequence into two short exact sequences

$$0 \rightarrow K_2 \rightarrow Q_1 \rightarrow K_1 \rightarrow 0$$

$$0 \rightarrow K_1 \rightarrow Q_0 \rightarrow N \rightarrow 0$$

which yield the following long exact sequences of $\text{Ext}(_, R)$:

$$0 \rightarrow \text{Hom}(K_1, R) \rightarrow \text{Hom}(Q_1, R) \rightarrow \text{Hom}(K_2, R) \rightarrow \text{Ext}^1(K_1, R) \rightarrow 0$$

$$0 \rightarrow \text{Hom}(N, R) \rightarrow \text{Hom}(Q_0, R) \rightarrow \text{Hom}(K_1, R) \rightarrow \text{Ext}^1(N, R) \rightarrow 0 \rightarrow \text{Ext}^1(K_1, R) \rightarrow 0.$$

The last zero arises since R has global dimension 1, and it implies $\text{Ext}^1(K_1, R) = 0$. The cokernel of the composition $\text{Hom}(Q_0, R) \rightarrow \text{Hom}(K_1, R) \rightarrow \text{Hom}(Q_1, R)$ is canonically isomorphic to an extension of the cokernels of the maps being composed, which is an extension of $\text{Ext}^1(N, R)$ by $\text{Hom}(K_2, R)$. The latter is projective, so the extension can (noncanonically) be taken to be a direct sum. \square

As one corollary, regardless of what projective resolution of $M(A)$ is used, the cokernel of the dual of its differential d_1 is $\text{Ext}^1(M(A), R) \cong M(A)_{\text{tors}}$ plus a projective module, so it differs from

$$M^*(E \setminus A) \cong \text{Ext}^1(M(A), R) \oplus \text{Hom}(P_2(A), R) \cong M(A)_{\text{tors}} \oplus P_2(A)^\vee$$

only up to projective summands.

Proof of Lemma 4.2. First of all, Lemma 3.1 implies that, given a fixed maximal flag of subsets $\{A_i\}$ of A , there is a unique choice of the modules I in each instance of (4.1), up to isomorphism. Therefore the isomorphism class of $P_2(A)$ is well-defined for each fixed flag.

We are done so long as every maximal flag of subsets yields the same projective module P_2 . One can obtain any maximal flag of subsets from any other by successive replacements of a segment $A_i \subsetneq A_i b \subseteq A_i bc$ with $A_i \subsetneq A_i c \subseteq A_i bc$, so it's sufficient to show that one such replacement doesn't alter $P_2(A)$. For any such replacement, there exists a commutative diagram as in axiom (M2).

$$\begin{array}{ccc} M(A_i) & \xrightarrow{f} & M(A_i b) \\ g \downarrow & & \downarrow g' \\ M(A_i c) & \xrightarrow{f'} & M(A_i bc) \end{array}$$

Whichever of the two flags of subsets is used, these two maps correspond to two steps like (4.1) in the filtration of $P(A)_\bullet$. In either case, the subquotient complex of $P(A)_\bullet$ formed from the extension formed of these two steps is a resolution of $\ker(M(A_i) \rightarrow M(A_i bc))$ like

$$0 \rightarrow K \rightarrow R^2 \xrightarrow{d} \ker(M(A_i) \rightarrow M(A_i bc)) \rightarrow 0$$

where the labelled map d may be chosen to be $(r, s) \mapsto rx + sy$ if x and y generate the kernels of f and g , respectively. It follows that K , and therefore the module $P_2(A)$, is isomorphic in the two cases.

Finally, by Lemma 4, since $M^*(E \setminus A)$ depends only on the isomorphism classes of $P_2(A)$ and $M(A)$ itself, it is well-defined. \square

Theorem 4.4. *If R is a Dedekind domain, and M is a matroid over R , then M^* is a matroid over R as well.*

Proof. Let $A \subseteq E$ and $b \in E \setminus A$. In the construction of $P(Ab)_\bullet$, choose a maximal flag of subsets ending in $\cdots \subseteq A \subseteq Ab$. The construction then provides an exact

sequence of complexes which, at the P_0 and P_1 terms, looks like

$$\begin{array}{ccccccc} 0 & \longrightarrow & P_1(A) & \longrightarrow & P_1(Ab) & \longrightarrow & R \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & P_0(A) & \longrightarrow & P_0(Ab) & \longrightarrow & 0 \end{array}$$

All these modules are projective, so dualizing all the maps preserves exactness: we have

$$(4.3) \quad \begin{array}{ccccccc} 0 & \longleftarrow & P_1(A)^\vee & \longleftarrow & P_1(Ab)^\vee & \longleftarrow & R^\vee \longleftarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longleftarrow & P_0(A)^\vee & \longleftarrow & P_0(Ab)^\vee & \longleftarrow & 0 \end{array}$$

This induces a map between the cokernels of the left two upward arrows, which is still surjective, and has kernel some quotient of R . That is, we have a surjection $M^*(E \setminus A) \leftarrow M^*(E \setminus (Ab))$ whose kernel is a cyclic module. These maps are exactly what is needed to establish axiom (M1) for M^* .

Now let $b, c \in E \setminus A$. Building off the maps in the pushout diagram assured by axiom (M2) for M , we get a commuting square of the maps among the modules P_1 constructed above.

$$\begin{array}{ccc} P_1(A) & \longrightarrow & P_1(Ab) \\ \downarrow & & \downarrow \\ P_1(Ac) & \longrightarrow & P_1(Abc) \end{array}$$

Each of these inclusions has cokernel R , and so the target splits as a direct sum. Regard the various complexes $P(\cdot)_\bullet$ as resolutions of kernels $\ker(P_0^\emptyset \rightarrow M(\cdot))$. Then, taking for example the top map, $P_1(A) \rightarrow P_1(Ab)$, we can identify $P_1(Ab)$ with $P_1(A) \oplus R$, where $P_1(A)$ maps to $\ker(P_0^\emptyset \rightarrow M(Ab))$ via its map to $\ker(P_0^\emptyset \rightarrow M(A))$, and R maps to $\ker(P_0^\emptyset \rightarrow M(Ab))$ by sending 1 to a lift of a generator of $\ker(M(A) \rightarrow M(Ab))$.

Now, this lift of a generator of $\ker(M(A) \rightarrow M(Ab))$ to P_0 is also a lift of a generator of $\ker(M(Ac) \rightarrow M(Abc))$. The same is true with the roles of b and c reversed. So in fact the whole square of maps can be split compatibly, as

$$\begin{array}{ccc} P_1(A) & \longrightarrow & P_1(A) \oplus R \\ \downarrow & & \downarrow (x,r) \mapsto (x,r,0) \\ P_1(A) \oplus R & \longrightarrow & P_1(A) \oplus R^2 \\ & & (x,r) \mapsto (x,0,r) \end{array}$$

Dualizing this square yields the square

$$(4.4) \quad \begin{array}{ccc} P_1(A)^\vee & \longleftarrow & P_1(A)^\vee \oplus R^\vee \\ \uparrow & & \uparrow \\ P_1(A)^\vee \oplus R^\vee & \longleftarrow & P_1(A) \oplus (R^\vee)^2 \end{array}$$

in which all the maps are projections onto summands, which is a pushout.

Finally, the square with which we are ultimately concerned

$$(4.5) \quad \begin{array}{ccc} M^*(E \setminus A) & \longleftarrow & M^*(E \setminus (Ab)) \\ \uparrow & & \uparrow \\ M^*(E \setminus (Ac)) & \longleftarrow & M^*(E \setminus (Abc)) \end{array}$$

is obtained by taking the quotient of each of the modules in square (4.4) by the image of the corresponding module $P_0(\cdot)^\vee$. In fact all the $P_0(\cdot)^\vee$ are isomorphic to $(P_0^\emptyset)^\vee$, compatibly. This remains a pushout by the universal property, as follows. Commuting maps from $M^*(E \setminus (Ab))$ and $M^*(E \setminus (Ac))$ to a module N lift to commuting maps to N from the upper-right and lower-left instances of $P_1(A)^\vee \oplus R^\vee$ in (4.4), whose kernels contain $(P_0^\emptyset)^\vee$. Since that square is a pushout, a map $P_1(A)^\vee \rightarrow N$ can be provided. The kernel of this map contains $(P_0^\emptyset)^\vee$ and so it descends to a map $M^*(E \setminus A) \rightarrow N$. Uniqueness can be argued similarly. We have thus established axiom (M2) for M^* . \square

We now state a corollary of Lemma 4, which we have postponed to here only so that “matroid over R ” could appear in the statement.

Corollary 4.5. *M^* is a full-rank matroid over R .*

Proof. The module $P_2(\emptyset)$ is trivial, and therefore $M^*(E) \cong M(\emptyset)_{\text{tors}}$ by Lemma 4. \square

Matroid duality over R has the properties expected of it.

Proposition 4.6. *If M is a matroid over a Dedekind domain R , then M is the direct sum of M^{**} and the projective empty matroid for $M(E)_{\text{proj}}$. In particular $M^{**} = M$ if M is full-rank.*

Proof of Proposition 4.6. Fix $A \subseteq E$, $b \in E \setminus A$, and a map $f : M(A) \rightarrow M(Ab)$ as provided by axiom (M1). Resuming the construction of diagram (4.3), the exact sequence of dual modules in that diagram (with R^\vee rewritten as R) fits together with the maps in the dual matroid M^* in the commuting square of exact sequences

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longleftarrow & J(A) & \longleftarrow & J(Ab) & \longleftarrow & I \longleftarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longleftarrow & P_1(A)^\vee & \longleftarrow & P_1(Ab)^\vee & \longleftarrow & R \longleftarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longleftarrow & M^*(E \setminus A) & \longleftarrow & M^*(E \setminus (Ab)) & \longleftarrow & R/I \longleftarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}.$$

Here I is some ideal of R , and for a set B , $J(B)$ is the image of $P_0(B)^\vee = (P_0^\emptyset)^\vee$ in $P_1(B)^\vee$. Since dualization is right exact, we have an exact sequence

$$0 \rightarrow \operatorname{Hom}(M(B), R) \xrightarrow{\delta(B)} (P_0^\emptyset)^\vee \rightarrow P_1(B)^\vee$$

so that $J(B) = \operatorname{coker} \delta(B)$. Therefore, I is the kernel of the natural map

$$\operatorname{coker} \delta(Ab) \rightarrow \operatorname{coker} \delta(A)$$

induced by the map $f^\vee : \operatorname{Hom}(M(Ab), R) \rightarrow \operatorname{Hom}(M(A), R)$. By the third isomorphism theorem, $I \cong \operatorname{coker}(f^\vee)$. Since the $\operatorname{Hom}(M(B), R)$ are projective modules, dualizing yields $I^\vee \cong \ker f^{\vee\vee}$, where $f^{\vee\vee}$ is the double dual map

$$f^{\vee\vee} : \operatorname{Hom}(\operatorname{Hom}(M(A), R), R) \rightarrow \operatorname{Hom}(\operatorname{Hom}(M(Ab), R), R)$$

induced by f . The source and target are respectively isomorphic to $M(A)_{\operatorname{proj}}$ and $M(Ab)_{\operatorname{proj}}$, so we have

$$(4.6) \quad I^\vee \oplus M(Ab)_{\operatorname{proj}} \cong M(A)_{\operatorname{proj}}.$$

Let $P^*(\cdot)_\bullet$ be the analogues of the resolutions $P(\cdot)_\bullet$ for M^* . By Lemma 4 we have

$$M^{**}(A) \cong \operatorname{Ext}^1(M^*(E \setminus A), R) \oplus P_2^*(E \setminus A)^\vee,$$

wherein the summand $\operatorname{Ext}^1(M^*(E \setminus A), R)$ is isomorphic to $M^*(E \setminus A)_{\operatorname{tors}}$. By Lemma 4 again this is $M(A)_{\operatorname{tors}}$, hence overall

$$M^{**}(A) \cong M(A)_{\operatorname{tors}} \oplus P_2^*(E \setminus A)^\vee$$

and, in parallel,

$$M^{**}(Ab) \cong M(Ab)_{\operatorname{tors}} \oplus P_2^*(E \setminus (Ab))^\vee.$$

By construction, the module $P_2^*(E \setminus A)$ is the direct sum of $P_2^*(E \setminus (Ab))$ and the ideal I from above, and the same is true of their duals. Therefore, by (4.6),

$$\begin{aligned} M^{**}(A) \oplus M(Ab)_{\operatorname{proj}} &\cong M(A)_{\operatorname{tors}} \oplus P_2^*(E \setminus (Ab))^\vee \oplus I^\vee \oplus M(Ab)_{\operatorname{proj}} \\ &\cong M(A)_{\operatorname{tors}} \oplus P_2^*(E \setminus (Ab))^\vee \oplus M(A)_{\operatorname{proj}} \\ &\cong M(A) \oplus P_2^*(E \setminus (Ab))^\vee \end{aligned}$$

whereas

$$(4.7) \quad \begin{aligned} M^{**}(Ab) \oplus M(Ab)_{\operatorname{proj}} &\cong M(Ab)_{\operatorname{tors}} \oplus P_2^*(E \setminus (Ab))^\vee \oplus M(Ab)_{\operatorname{proj}} \\ &\cong M(Ab) \oplus P_2^*(E \setminus (Ab))^\vee. \end{aligned}$$

In the algebraic K-theory of R , these two isomorphisms say

$$(4.8) \quad [M(A)] - [M^{**}(A)] = [M(Ab)_{\operatorname{proj}}] - [P_2^*(E \setminus (Ab))^\vee] = [M(Ab)] - [M^{**}(Ab)].$$

By Corollary 4.5, $M^{**}(E)_{\operatorname{proj}}$ is trivial, and therefore $P_2^*(\emptyset)^\vee$ is trivial in equation (4.7) and $M^{**}(E) = M(E)_{\operatorname{tors}}$, that is, $M(E) = M^{**}(E) \oplus M(E)_{\operatorname{proj}}$. In K-theory this says $[M(E)] - [M^{**}(E)] = [M(E)_{\operatorname{proj}}]$. By repeated application of equation (4.8), $[M(A)] - [M^{**}(A)]$ is equal to $[M(E)_{\operatorname{proj}}]$ for any A . But since $M(A)$ and $M^{**}(A)$ have the same torsion submodule, Proposition 3.3 implies that

$$M(A) = M^{**}(A) \oplus M(E)_{\operatorname{proj}},$$

which was the claim. \square

Proposition 4.7. *Let M be a matroid over R . The element*

$$\begin{aligned} \text{cl}(M) &\doteq [M(A)_{\text{proj}}] + [M^*(E \setminus A)_{\text{proj}}] + [M(A)_{\text{tors}}] \\ &= [M(A)] + [M^*(E \setminus A)] - [M(A)_{\text{tors}}] \end{aligned}$$

of $\text{Pic}(R)$ is independent of the choice of $A \subseteq E$.

In particular, if A is an independent set (or, more strongly, a basis) of the generic matroid of M , then by Corollary 4.5 we have $[M(A)] = \text{cl}(M)$.

Proof. Given a set $A \subseteq E$, let $\text{cl}(M)(A)$ be the value of $\text{cl}(M)$ computed using that given choice of A . It is enough to show that, for each $A \subseteq E$ and $b \in E \setminus A$, $\text{cl}(M)(A)$ equals $\text{cl}(M)(Ab)$.

Given A and b , it is true of exactly one of the two dual maps $M(A) \rightarrow M(Ab)$ and $M^*(E \setminus (Ab)) \rightarrow M^*(E \setminus A)$ provided by axiom (M1) that the rank of the target is one less than the rank of the source. In the other map, these two ranks are equal.

If the first map has this rank drop, then its kernel must be isomorphic to R , and the exact sequence

$$0 \rightarrow R \rightarrow M(A) \rightarrow M(Ab) \rightarrow 0$$

implies that $[M(A)] = [M(A)_{\text{proj}}] + [M(A)_{\text{tors}}]$ equals $[M(Ab)] = [M(Ab)_{\text{proj}}] + [M(Ab)_{\text{tors}}]$. Since the second map has no rank drop, its kernel is contained in the torsion submodule of its source, so $[M(E \setminus A)_{\text{proj}}]$ equals $[M(E \setminus (Ab))_{\text{proj}}]$. Adding these equalities, we have $\text{cl}(M)(A) = \text{cl}(M)(Ab)$.

If instead the second map has the rank drop, then the same argument shows that $\text{cl}(M)(A) = \text{cl}(M)(Ab)$ after exchanging M for M^* and sets A for their complements $E \setminus A$, and using the fact that $M(A)_{\text{tors}} \cong M^*(E \setminus A)_{\text{tors}}$. \square

Recall that $K_0(R) = \mathbb{Z} \oplus \text{Pic}(R)$. Let $\sigma : K_0(R) \rightarrow K_0(R)$ be the involution acting as the identity on the summand \mathbb{Z} and negation on the summand $\text{Pic}(R)$.

Corollary 4.8. *If M is a matroid over R , then $M^*(E \setminus A)$ is the module whose torsion part is $M^*(E \setminus A)_{\text{tors}} \cong M(A)_{\text{tors}}$ and whose projective part is determined by the equality*

$$(4.9) \quad [M^*(E \setminus A)_{\text{proj}}] = \sigma([M(A)] + |A| \cdot [R] - [M(\emptyset)])$$

in $K_0(R)$.

Note that, over a field, equation (4.9) specializes to the formula for dualizing rank functions familiar from the matroid setting,

$$\text{cork}_M^*(E \setminus A) = \text{cork}_M(A) + |A| - r$$

where r is the rank of M .

Proof. The assertion on the torsion parts is the discussion after .

As for the projective part, we treat the summands of $K_0(R) = \mathbb{Z} \oplus \text{Pic}(R)$ separately. In the $\text{Pic}(R)$ summand, Proposition 4.7 implies that

$$[M^*(E \setminus A)_{\text{proj}}] + [M(A)] = \text{cl}(M) = [M^*(E)_{\text{proj}}] + [M(\emptyset)],$$

which becomes the $\text{Pic}(R)$ part of (4.9) on noting that M^* is full-rank so $M^*(E)_{\text{proj}}$ is trivial.

Regarding rank, consider again the ideal $I \cong \text{coker}(f^\vee)$ in the proof of Proposition 4.6. Since $^\vee$ preserves rank and f^\vee is an injection, we have

$$\text{rank } I = \text{rank } M(A) - \text{rank } M(Ab).$$

As well,

$$\text{rank } M^*(E \setminus Ab) - \text{rank } M^*(E \setminus A) = \text{rank } R/I = 1 - \text{rank } I.$$

By induction on the size of $E \setminus A$, we get that $\text{rank } M(A) - \text{rank } M^*(E \setminus A) + |A|$ is constant, and thus always equal to its value $\text{rank } M(\emptyset)$ taken when $A = \emptyset$. This proves the part of (4.9) in the \mathbb{Z} summand. \square

Proposition 4.9. *If M and M' are full-rank matroids over a Dedekind domain R , then*

- (a) $(M \oplus M')^* = M^* \oplus M'^*$.
- (b) $(M/i)^* = M^* \setminus i$ if i is not a generic loop, and $(M \setminus i)^* = M^*/i$ if i is not a generic coloop.
- (c) Let $f : R \rightarrow S$ be a flat map to a Dedekind domain S . Then $(M \otimes S)^* = M^* \otimes S$ (as matroids over S).
- (d) Let $f : R \rightarrow S$ be the quotient by a maximal ideal. Then, again, $(M \otimes S)^* = M^* \otimes S$.

Proof. Since direct sums, deletions and contractions, and base changes are computed one module at a time, these are all fairly straightforward to check given Corollary 4.8.

Using that corollary, part (a) follows because \oplus commutes with taking the torsion part, and everything on the right side of equation (4.9) is additive when applied to $(M \oplus M')^*((E \amalg E') \setminus (A \amalg A'))$.

For part (b), the torsion parts again take care of themselves. For the projective parts, in the first case, i not being a generic loop ensures that $M(\emptyset)$ is isomorphic to $(M \setminus i)(\emptyset) \oplus R$, and this empty summand of R cancels the extra $[R]$ on the right side of (4.9) arising from the discrepancy between the contraction and the deletion.

For part (c), to begin, we have $(M \otimes S)^*(A)_{\text{tors}} = (M(E \setminus A) \otimes S)_{\text{tors}}$ directly. On the other hand, since projective modules remain projective under $- \otimes S$, we find that $(M^*(A) \otimes S)_{\text{tors}}$ equals $(M^*(A)_{\text{tors}} \otimes S)_{\text{tors}}$, which in turn is $(M(E \setminus A)_{\text{tors}} \otimes S)_{\text{tors}} = (M(E \setminus A) \otimes S)_{\text{tors}}$. So the torsion parts agree.

As for the projective parts, because f is flat, the induced homomorphism $f_* : K_0(R) \rightarrow K_0(S)$ is given simply by $f_*[N] = [N \otimes S]$. Also, torsion modules remain torsion on tensoring with S , so that the operations $- \otimes S$ and $-_{\text{proj}}$ commute. Hence, using (4.9),

$$[(M \otimes S)^*(A)_{\text{proj}}] = \sigma([M(A) \otimes S] + |A| \cdot [S] - [M(\emptyset) \otimes S])$$

equals

$$\begin{aligned} [(M^* \otimes S)(A)_{\text{proj}}] &= [M^*(A)_{\text{proj}} \otimes S] = f_*[M^*(A)_{\text{proj}}] \\ &= f_*\sigma([M(A)] + |A| \cdot [R] - [M(\emptyset)]). \end{aligned}$$

For part (d), $M \otimes S$ is a classical matroid, over a field, and so we need only check that the corank functions of $(M \otimes S)^*$ and $M^* \otimes S$ are equal. Let I be the maximal ideal such that $S = R/I$. For a f.g. R -module N , the S -dimension of $N \otimes S$

is the rank of N_{proj} plus $\dim_S \text{Tor}_1^R(N, S)$; the latter summand is the number of indecomposable summands of N isomorphic to R/I^n for some n . Now,

$$\text{cork}_{(M \otimes S)^*}(E \setminus A) = \text{cork}_{M \otimes S}(A) + |A| - r$$

where r is the generic rank of $M \otimes S$. The term $\text{cork}_{M \otimes S}(A)$ is computed as described just above, for $N = M(A)$: we get $\text{rank } M(A) + \dim_S \text{Tor}_1^R(M(A), S)$. On the other hand, we know that $M^*(E \setminus A)$ has the same projective part as $M(A)$; this means $\text{Tor}_1^R(M^*(E \setminus A), S) = \text{Tor}_1^R(M(A), S)$. And the rank of the module $M^*(E \setminus A)$ is $\text{rank } M(A) + |A| - \text{rank } M(\emptyset) = \text{rank } M(A) + |A| - r$, by (4.9). Therefore the dimension of $M^*(E \setminus A)$ is

$$\text{rank } M(A) + |A| - r + \dim_S \text{Tor}_1^R(M^*(E \setminus A), S),$$

which agrees with $\text{cork}_{(M \otimes S)^*}(E \setminus A)$ as required. \square

Our last proposition on duality gives a generalization of the classical Gale duality of vector configurations. It was prefigured by the construction of the dual of a representable arithmetic matroid in [6], in which the matrix transpose operation used to construct the Gale dual can be taken to correspond to dualizing our differential d_1 .

Proposition 4.10. *If a matroid M over a Dedekind domain R is representable, then its dual M^* is representable too.*

Proof. Let M be represented by the vector configuration $(x_a : a \in E)$ within $M(\emptyset)$. Fix lifts of these vectors to vectors $(\tilde{x}_a : a \in E)$ within P_0^\emptyset .

For a set $A \subseteq E$ and an element $a \in A$, the map $d_1^A : P_1^\emptyset \oplus R^A \rightarrow P_0^\emptyset$ appearing in the resolution $P(A)_\bullet$ satisfies $d_1(0, e_a) = \tilde{x}_a$. Thus each of these maps arises from the map $d_1^E : P_1^\emptyset \oplus R^E \rightarrow P_0^\emptyset$ in the complex $P(E)_\bullet$ by restricting it to $P_1^\emptyset \oplus R^A \subseteq P_1^\emptyset \oplus R^E$. Let us dualize, and write $\{e^a : a \in E\}$ for the dual standard basis of $(R^E)^\vee$. The map $(d_1^A)^\vee$ factors as $q_A \circ (d_1^E)^\vee$, where $q_A : (P_1^\emptyset \oplus R^E)^\vee \rightarrow (P_1^\emptyset \oplus R^A)^\vee$ is the quotient map by the submodule $\langle e^a : a \in E \setminus A \rangle$. Hence, $M^*(E \setminus A) = \text{coker}((d_1^A)^\vee)$ is the quotient of $M^*(\emptyset) = \text{coker}((d_1^E)^\vee)$ by the submodule generated by the images of the e^a for $a \in E \setminus A$. But we have now exactly described a vector configuration representing M^* : the ambient module is $M^*(\emptyset) = \text{coker}((d_1^E)^\vee)$, and the vector labelled by a is the image of e^a . \square

5. STRUCTURE OF MATROIDS OVER A DVR

In this section and the next we record some structure theorems for matroids over R in terms of structure theorems for the modules over R themselves. Our analysis of general Dedekind domains in the next section will make much use of base changing to localizations of R , so we begin here with the local case.

For the whole of this section, R will be a DVR with maximal ideal \mathfrak{m} . We first recall the structure theory of f.g. R -modules: any indecomposable f.g. R -module is isomorphic to either R or R/\mathfrak{m}^n for some integer $n \geq 1$. We will sometimes formally subsume R into the latter family by writing it as R/\mathfrak{m}^∞ . So, if N is a f.g. R -module and $i \geq 1$ is an integer, define

$$d_i(N) \doteq \dim_{R/\mathfrak{m}}(\mathfrak{m}^{i-1}N/\mathfrak{m}^iN),$$

and

$$d_{\leq i}(N) \doteq \sum_{j=1}^i d_j(N) = \dim_{R/\mathfrak{m}}(N/\mathfrak{m}^i N),$$

and for convenience $d_i(N) = d_{\leq i}(N) = 0$ if $i \leq 0$. Let $d_{\bullet}(N)$ denote the infinite sequence of these. We have

$$d_i(R/\mathfrak{m}^n) = \begin{cases} 1 & 0 < i \leq n \\ 0 & i > n \end{cases},$$

where n may be ∞ . The following is a quick consequence.

Proposition 5.1. *Isomorphism types of f.g. R -modules are in bijection with non-increasing infinite sequences d_{\bullet} of nonnegative integers indexed by positive integers, the bijection being given by*

$$N \longleftrightarrow d_{\bullet}(N).$$

This bijection permits a straightforward identification of those isomorphism classes of modules which permit maps satisfying axiom (M1).

Proposition 5.2. *Let N and N' be f.g. R -modules. There exists a surjection $\phi : N \rightarrow N'$ with cyclic kernel if and only if*

$$d_i(\phi) \doteq d_i(N) - d_i(N')$$

equals 0 or 1 for each $i \geq 1$.

We can also easily extract the $d_i(\phi)$.

Corollary 5.3. *Let $\{e_{\alpha}\}$ be a minimal set of generators for an f.g. R -module N , and suppose e_{α} generates a summand isomorphic to $R/\mathfrak{m}^{\ell_{\alpha}}$, wherein ℓ_{α} may be ∞ . Let $x = \sum x_{\alpha}e_{\alpha}$ be an element of N , and ϕ the canonical map $N \rightarrow N/\langle x \rangle$. Then $d_{\bullet}(\phi)$ is the lexicographically least sequence d_{\bullet} such that for every α ,*

$$(5.1) \quad \#\{i \leq \ell_{\alpha} : d_i = 0\} \leq \dim_{R/\mathfrak{m}}(\langle e_{\alpha} \rangle / \langle x_{\alpha}e_{\alpha} \rangle).$$

When ℓ_{α} is finite, condition (5.1) is equivalent to

$$d_{\leq \ell_{\alpha}} \geq \dim_{R/\mathfrak{m}}(\langle x_{\alpha}e_{\alpha} \rangle).$$

In the case that N and N' have finite length, Proposition 5.2 follows from facts about the Hall algebra [14]. Indeed, it is equivalent that N have finite length and that $d_i(N)$ stabilize to 0 for $i \gg 0$. In this case d_i is a partition, and its conjugate partition is the one usually used to label N . For a cyclic module, this conjugate partition has a single row. Then, under the specialization taking the Hall polynomials to the Littlewood-Richardson coefficients, Proposition 5.2 is a consequence of the Pieri rule. (Taking this further, our foundational Lemma 3.1 is essentially the statement that all coefficients in the Pieri rule are equal to 1.)

We include a proof of the proposition nonetheless, both because we do not require finite length and because we reuse its framework in proving Corollary 5.3.

Proof. Necessity. Let $\langle x \rangle$ be the cyclic kernel of $N \rightarrow N'$, for $x \in N$. The kernel of the induced surjection $N \otimes R/\mathfrak{m}^n \rightarrow N' \otimes R/\mathfrak{m}^n$ is

$$K_n = \langle x \rangle / (\langle x \rangle \cap \mathfrak{m}^n N).$$

The dimensions over R/\mathfrak{m} of these three modules are related by

$$d_{\leq n}(N) - d_{\leq n}(N') = \dim_{R/\mathfrak{m}} K_n$$

and, by subtracting two such relations,

$$d_n(N) - d_n(N') = \dim_{R/\mathfrak{m}} K_n - \dim_{R/\mathfrak{m}} K_{n-1}.$$

It is clear by definition that the K_n are an increasing sequence of modules, so that $\dim_{R/\mathfrak{m}} K_n - \dim_{R/\mathfrak{m}} K_{n-1}$ is nonnegative. On the other hand,

$$(\langle x \rangle \cap \mathfrak{m}^{n-1}N) / (\langle x \rangle \cap \mathfrak{m}^nN)$$

has length at most 1, since if $\mathfrak{m}^i x \subseteq \mathfrak{m}^{n-1}N$ then $\mathfrak{m}^{i+1}x \subseteq \mathfrak{m}^nN$. But this length is $\dim_{R/\mathfrak{m}} K_n - \dim_{R/\mathfrak{m}} K_{n-1}$, which is thus at most 1.

Sufficiency. Given N and an infinite list $\delta_i \in \{0, 1\}$ such that $d_i(N) - \delta_i$ is also a nonincreasing sequence of naturals, equal therefore to $d_i(N')$ for a module N' , we wish to construct $x \in N$ so that $N/\langle x \rangle \cong N'$.

Let I be the set of indices i for which $\delta_i = 1$ and $\delta_{i+1} = 0$; also include in I the symbol ∞ if $\delta_i = 1$ for all sufficiently large i . For each $i \in I$, there is a summand isomorphic to R/\mathfrak{m}^i in N . Splitting off one module of each of these isomorphism classes, we can make the identification

$$N = \bigoplus_{i \in I} R/\mathfrak{m}^i \oplus P$$

for some module P , and let $e_i : i \in I$ be generators of the summands other than P . Let $t \in R$ be a generator of \mathfrak{m} , and define

$$x = \sum_{k \in I} t^{k - \delta_{\leq k}} e_k,$$

where as expected $\delta_{\leq k}$ means $\sum_{i=1}^k \delta_i$.

The module P will remain as a summand in $N/\langle x \rangle$, and we may restrict attention to the remaining summand, call it Q . Towards describing it, define the elements

$$\tilde{e}_i = \sum_{k \in I, k \geq i} t^{(k - \delta_{\leq k}) - (i - \delta_{\leq i})} e_k \in N.$$

Fix for the moment some $i \in I$. Let $j = j(i)$ be the greatest index less than i such that $\delta_j = 0$ and $\delta_{j+1} = 1$, or if there is no such index let $j = 0$. Then we have

$$t^j \tilde{e}_i = t^{\delta_{\leq j}} x.$$

This is because $j - \delta_{\leq j} = i - \delta_{\leq i}$ by the definition of i , so that the coefficients of e_k agree for all $k \geq i$; for $k < i$, however, we also have $k < j$ and thus $k - \delta_{\leq k} + \delta_{\leq j} \geq k$, so that the coefficient in $t^j \tilde{e}_i$ of e_k is zero. Therefore, $t^j \tilde{e}_i$ equals zero in $N/\langle x \rangle$.

However, if some R -linear combination $y = \sum_{i \in I} r_i \tilde{e}_i \in N$ is zero in $N/\langle x \rangle$, then $r_i \in \mathfrak{m}^{j(i)}N$ for each i . Otherwise, write $y = sx$. Let i be minimal so that $r_i \notin \mathfrak{m}^{j(i)}N$, and let $j = j(i)$. If y is expanded in terms of the e_k , then the least k such that e_k has a nonzero coefficient is $k = i$. Let i' be the greatest element of I less than i . Since the coefficient of $e_{i'}$ in y is zero, the \mathfrak{m} -valuation of s must be greater than or equal to $i' - (i' - \delta_{\leq i'}) = \delta_{\leq i'} = \delta_{\leq j}$, in view of the definition of x . (Or, if there is no element of I less than i , then consideration of the coefficient of e_i in x yields the same conclusion.) But then the \mathfrak{m} -valuation of the coefficient of

e_i in y is greater than or equal to $(i - \delta_{\leq i}) + \delta_{\leq j} = j$, contradicting our assumption on i .

It follows that the R -module generated by the \tilde{e}_i is isomorphic to

$$\bigoplus_{i \in I} R/\mathfrak{m}^{j(i)},$$

wherein $\{j(i) : i \in I\}$ is the set of all indices j for which $\delta_j = 0$ and $\delta_{j+1} = 1$. The elements \tilde{e}_i in fact generate Q , by a triangularity argument between the \tilde{e}_i and the e_i . We conclude that the sequences $d_i(N) - d_i(N')$ and δ_i are equal. \square

Proof of Corollary 5.3. Let $\nu_\alpha = \dim_{R/\mathfrak{m}}(\langle e_\alpha \rangle / \langle x_\alpha e_\alpha \rangle)$; this is the maximum of ℓ_α and the \mathfrak{m} -valuation of x_α . Suppose first that $x_\alpha = 0$ for all α except for a single list $A = \{\alpha_1, \dots, \alpha_{|A|}\}$ such that both (ν_{α_i}) and $(\ell_{\alpha_i} - \nu_{\alpha_i})$ are strictly increasing sequences. To avoid proliferation of subscripts we will write $\nu_i \doteq \nu_{\alpha_i}$ and $\ell_i \doteq \ell_{\alpha_i}$.

The condition (5.1) is vacuous when $x_\alpha = 0$. The sequence d_\bullet that we obtain from (5.1) for the $\alpha \in A$ is

$$0^{\nu_1} 1^{\ell_1 - \nu_1} 0^{\nu_2 - \nu_1} 1^{\ell_2 - \nu_2 - \ell_1 + \nu_1} 0^{\nu_3 - \nu_2} 1^{\ell_3 - \nu_3 - \ell_2 + \nu_2} \dots,$$

exponents indicating repetition. For this sequence, if the sufficiency argument of Proposition 5.2 is run with the same choice of generators $\{e_\alpha\}$, the element x produced to generate the kernel is the same one we have chosen here (up to automorphisms of the cyclic summands $\langle e_\alpha \rangle$). So the corollary is proven in this case.

Now suppose two indices α and α' are such that

$$(5.2) \quad \nu_\alpha \leq \nu_{\alpha'} \quad \text{and} \quad \ell_\alpha - \nu_\alpha \geq \ell_{\alpha'} - \nu_{\alpha'}.$$

The inequality (5.1) holds if and only if the $(\nu_\alpha + 1)$ th 0 of d_\bullet , if any, follows at least $\ell_\alpha - \nu_\alpha$ 1s. Hence, (5.1) for α' is implied by (5.1) for α , and thus α' is irrelevant for computing d_\bullet . Moreover, inequalities (5.2) ensure that we may change our basis for N by adding a multiple of e'_α to e_α , yielding another generator \tilde{e}_α of $\langle e_\alpha \rangle$, so that

$$x_\alpha e_\alpha + x_{\alpha'} e_{\alpha'} = \tilde{x}_\alpha \tilde{e}_\alpha$$

for some \tilde{x}_α with the same \mathfrak{m} -valuation as x_α .

By repeatedly making such changes of basis, we may, with no changes to the sequence d_\bullet that will be computed, assume that $x_\alpha = 0$ for all α except for a set no two of whose members α, α' satisfy (5.2). But such a set may be ordered so that (ν_{α_i}) and $(\ell_{\alpha_i} - \nu_{\alpha_i})$ are both strictly increasing, and this reduces to the first case. \square

Having control over axiom (M1), we turn to axiom (M2).

Proposition 5.4. *Let $M(\emptyset)$, $M(1)$, $M(2)$, and $M(12)$ be f.g. R -modules. There exist four surjections with cyclic kernels forming a pushout square*

$$\begin{array}{ccc} M(\emptyset) & \xrightarrow{\phi} & M(1) \\ \psi \downarrow & \lrcorner & \downarrow \psi' \\ M(2) & \xrightarrow[\phi']{} & M(12) \end{array}$$

if and only if

- (L1) the source and target of each map satisfy Proposition 5.2;
 (L2a) for each $n \geq 1$,

$$d_{\leq n}(M(\emptyset)) - d_{\leq n}(M(1)) - d_{\leq n}(M(2)) + d_{\leq n}(M(12)) \geq 0;$$

 (L2b) for any $n \geq 1$ such that $d_n(M(1)) \neq d_n(M(2))$, equality holds above:

$$d_{\leq n}(M(\emptyset)) - d_{\leq n}(M(1)) - d_{\leq n}(M(2)) + d_{\leq n}(M(12)) = 0.$$

The numbering of these conditions is chosen to agree with the numbering of the axioms for a quasi-arithmetic matroid in Corollary 6.3.

Condition (L2a) has a standard name: it asserts that $A \mapsto d_{\leq n}(M(A))$ is a *supermodular* function. Supermodular functions are less familiar than their opposite number, submodular functions: a supermodular function is the negative of a submodular one (that is, the inequalities are reversed).

Proof. Necessity. Condition (L1) is clear from the fact that axiom (M2) implies axiom (M1).

Tensoring the matroid M with R/\mathfrak{m}^n gives a matroid $M' \doteq M \otimes (R/\mathfrak{m}^n)$ over that ring. All of its modules are of finite length. Now regard these modules $M'(A)$ as R/\mathfrak{m} -vector spaces. The maps $M'(A) \rightarrow M'(Ab)$ given by (M1) remain surjective, and the pushout diagrams in (M2) remain pushouts, since surjectivity and pushout-hood can be checked set-theoretically. Accordingly, M' can be interpreted as a polymatroid over R/\mathfrak{m} , that is, a classical polymatroid. The corank function of a polymatroid is supermodular, and this is Condition (L2a).

As for condition (L2b), suppose that the inequality of (L2a) were strict. Let the kernel of ϕ be $\langle x \rangle$, and the kernel of ψ be $\langle y \rangle$, so that the kernel of the composite $\psi' \circ \phi = \phi' \circ \psi$ is $\langle x, y \rangle$. So our assumption is

$$\dim \langle x \rangle / (\langle x \rangle \cap \mathfrak{m}^n N) + \dim \langle y \rangle / (\langle y \rangle \cap \mathfrak{m}^n N) > \dim \langle x, y \rangle / (\langle x, y \rangle \cap \mathfrak{m}^n N)$$

where all dimensions are over R/\mathfrak{m} . (Note that the non-strict version of this inequality manifestly holds, providing another proof of (L2a).) That is, there exist $r, s \in R$ such that $sy - rx \in \mathfrak{m}^n N$, but neither rx nor sy is in $\mathfrak{m}^n N$.

Now, suppose that $d_n(M(\emptyset)) - d_n(M(1)) = 1$. By the proof of Proposition 5.2, the module $(\langle x \rangle \cap \mathfrak{m}^{n-1} N) / (\langle x \rangle \cap \mathfrak{m}^n N)$ is nontrivial, i.e. there exists $q \in R$ so that

$$qx \in \mathfrak{m}^{n-1} N \setminus \mathfrak{m}^n N.$$

Because $qx \in \mathfrak{m}^{n-1} N$ and $rx \notin \mathfrak{m}^n N$, we have that r divides q in R , say $r = pq$. Then

$$psy - qx = p(sy - rx) \in \mathfrak{m}^n N$$

and by adding, we get $psy \in \mathfrak{m}^{n-1} N \setminus \mathfrak{m}^n N$, which implies that $d_n(M(\emptyset)) - d_n(M(2)) = 1$. Of course the same holds with the roles of 1 and 2 in the ground set reversed, so that $d_n(M(1)) = d_n(M(2))$. By contradiction, (L2b) is proved.

Sufficiency. Suppose the modules $M(A)$ satisfy (L1), (L2a), (L2b). By (L1), if f is one of the maps in the pushout, the sequence $d_{\bullet}(f)$ has elements drawn from $\{0, 1\}$. Let $I(f)$ be the set of positions i such that $d_i(f) = 1$ and $d_{i+1}(f) = 0$, together with ∞ if d_{\bullet} stabilizes at 1. We are assured of the existence of various simultaneous cyclic summands of $M(\emptyset)$ (i.e. all participating in a single direct sum decomposition), for which we may choose generators as follows: a generator e_i with $\langle e_i \rangle = R/\mathfrak{m}^i$ for each $i \in I(\phi) \cup I(\psi)$, and a generator ε_i (distinct from e_i) with $\langle \varepsilon_i \rangle = R/\mathfrak{m}^i$ for each $i \in I(\phi') \cap I(\psi')$.

Let $t \in R$ be a generator of \mathfrak{m} and define two elements x and y in $M(\emptyset)$ by

$$\begin{aligned} x &= \sum_{i \in I(\phi)} t^{i-d_{\leq i}(\phi)} e_i \\ y &= \sum_{i \in I(\psi)} t^{i-d_{\leq i}(\psi)} e_i + \sum_{i \in I(\psi') \setminus (I(\psi) \setminus I(\phi))} t^{i-d_{\leq i}(\psi')} \varepsilon_i. \end{aligned}$$

Let ϕ be the quotient map on $M(\emptyset)$ by $\langle x \rangle$, ψ the quotient map by $\langle y \rangle$, and ϕ' and ψ' the maps completing this to a pushout. We must check that the images of these maps, i.e. the quotients $M(\emptyset)/\langle x \rangle$, $M(\emptyset)/\langle y \rangle$, and $M(\emptyset)/\langle x, y \rangle$ have the isomorphism type they should.

The element x is the same one we constructed in the proof of Proposition 5.2, so $M(\emptyset)/\langle x \rangle \cong M(1)$. Next consider y . By (L2a), we have that $d_{\leq i}(\psi') \leq d_{\leq i}(\psi)$ for any i , and therefore $i - d_{\leq i}(\psi') \geq i - d_{\leq i}(\psi)$. Among the first i entries of $d_{\bullet}(\psi)$ there are exactly $i - d_{\leq i}(\psi)$ zeroes, and hence at most $i - d_{\leq i}(\psi')$ of them. Therefore, by Corollary 5.3, the presence of the second sum in the definition of y doesn't affect the isomorphism type of $M(\emptyset)/\langle y \rangle$, and parallel to the x case we have $M(\emptyset)/\langle y \rangle \cong M(2)$.

Lastly, we wish to show that $M(1)$ modulo the image $\phi(y)$ is isomorphic to $M(12)$. As our set of generators of $M(1)$ we will use the images of the generators we have defined for $M(\emptyset)$, except replacing the e_i for $i \in I(\phi)$ with the \tilde{e}_i defined in the proof of Proposition 5.2. From the definition of the \tilde{e}_i , we obtain the following base change formulae: for $i \in I(\phi)$, if i' is the minimum element of $I(\phi)$ exceeding i , and j is the unique index with $i < j < i'$ such that $d_j(\phi) = 0$ and $d_{j+1}(\phi) = 1$, then

$$\begin{aligned} e_i &= \tilde{e}_i - t^{(i'-\delta_{\leq i'})-(i-\delta_{\leq i})} \tilde{e}_{i'} \\ &= \tilde{e}_i - t^{j-i} \tilde{e}_{i'}. \end{aligned}$$

If there is no i' then $e_i = \tilde{e}_i$.

If i is in $I(\psi')$, the latter sum in the definition of y includes a term $t^{i-d_{\leq i}(\psi')} \varepsilon_i$, which is also a term in our expansion of $\phi(y)$, unless $i \in I(\psi)$ and $i \notin I(\phi)$. The latter noninclusion implies that the image of e_i is still one of the elements in our set of generators for $M(1)$. We also have $d_i(\psi) = 1$ and $d_{i+1}(\psi) = 0$ while either $d_i(\phi) = 0$ or $d_{i+1}(\phi) = 1$. In the former case, (L2b) immediately implies $d_{\leq i}(\psi) = d_{\leq i}(\psi')$; in the latter case, it implies $d_{\leq i+1}(\psi) = d_{\leq i+1}(\psi')$, from which the statement with i in place of $i+1$ follows. Therefore, in either case the former sum in the definition of y includes a term $t^{i-d_{\leq i}(\psi)} e_i = t^{i-d_{\leq i}(\psi')} e_i$, which is also a term in $\phi(y)$. These terms of $\phi(y)$ establish inequalities on the sequence $d_{\bullet}(\psi)$, of the sort described in Corollary 5.3, which $d_n(M(1)) - d_n(M(12))$ is the lexicographically first sequence satisfying. To complete the proof, we need only check that the terms of $\phi(y)$ we have not yet accounted for introduce no incompatible inequalities (much like we just checked for y).

The remaining terms of $\phi(y)$ are those corresponding to terms of y in the first sum which we have not treated yet, in $\langle e_i \rangle$ for $i \in I(\psi) \setminus I(\psi')$. Now, any such i is in $I(\phi)$. To justify this, we have $(d_i(\psi), d_{i+1}(\psi)) = (1, 0)$ while $(d_i(\psi'), d_{i+1}(\psi'))$ is some other two-bit sequence. The easiest case is $(d_i(\psi'), d_{i+1}(\psi')) = (0, 1)$. Then $(d_i(\phi), d_{i+1}(\phi)) = (1, 0)$ follows because

$$d_n(\phi) + d_n(\psi') = d_n(M(\emptyset)) - d_n(M(12)) = d_n(\psi) + d_n(\phi')$$

for all n , and the d_n are in $\{0, 1\}$. Next suppose $(d_i(\psi'), d_{i+1}(\psi')) = (0, 0)$. Then $d_i(\phi) = 1$ and $d_i(\phi') = 0$. Since property (L2a) at $i - 1$, the equality in this property must be strict at i , and therefore it would contradict property (L2b) if $d_{i+1}(\phi)$ were 1. The argument is similar with 0 and 1 exchanged and sequences reversed if $(d_i(\psi'), d_{i+1}(\psi')) = (1, 1)$. Therefore $i \in I(\phi)$ as claimed. It follows that our remaining terms of $\phi(y)$, are terms containing generators of the form \tilde{e}_i .

For any index $i \in I(\psi) \cap I(\phi)$, let $k \leq i$ be minimal and $\ell \geq i$ maximal (possibly ∞) such that $d_j(\phi) = d_j(\psi)$ for all $k < j \leq \ell$. Summing over this range, we have

$$\begin{aligned} \sum_{\substack{i \in I(\psi) \cap I(\phi) \\ k < i \leq \ell}} t^{i-d_{\leq i}(\psi)} e_i &= \sum_{\substack{i \in I(\psi) \cap I(\phi) \\ k < i \leq \ell}} t^{i-d_{\leq i}(\psi)} \tilde{e}_i - t^{i'-d_{\leq i'}(\psi)} e_{i'} \\ &= t^{i_0-d_{\leq i_0}(\psi)} \tilde{e}_{i_0} - t^{i_1'-d_{\leq i_1'}(\psi)} e_{i_1'}, \end{aligned}$$

where i_0 is the least element of $I(\psi) \cap I(\phi) \cap (k, \ell]$ and i_1 is the greatest; if i_1' doesn't exist, the term containing it above simply drops. Property (L2b) applies at $n = k$, so by property (L2a), we must have $d_i(\psi') = 0$ for all $k < i \leq i_0$. Therefore the inequality of Corollary 5.3 for the term $t^{i_0-d_{\leq i_0}(\psi)} \tilde{e}_{i_0}$ is satisfied by the sequence $d_n(M(1)) - d_n(M(12))$. Likewise, if there is an i_1' , then ℓ is finite, and property (L2b) applies at $n = \ell$, whence property (L2a) implies that $d_i(\psi') = 1$ for all $i_1 < i \leq \ell$. Therefore the inequality of Corollary 5.3 for the term $t^{i_1'-d_{\leq i_1'}(\psi)} e_{i_1'}$ is satisfied by the sequence $d_n(M(1)) - d_n(M(12))$ as well. This accounts for the last of the terms of $\phi(y)$, and at last we conclude by that corollary that $M(1)/\langle \phi(y) \rangle \cong M(12)$, as desired. \square

By the time we come to three-element matroids over R , there are already non-trivial conditions on the supermodular functions $d_{\leq n}$.

Proposition 5.5. *Let M be a matroid over R on the ground set $[3]$, and let n be a natural or ∞ . Then, among the three quantities*

$$d_{\leq n}(M(1)) + d_{\leq n}(M(23)), d_{\leq n}(M(2)) + d_{\leq n}(M(13)), d_{\leq n}(M(3)) + d_{\leq n}(M(12)),$$

the minimum is achieved at least twice.

Proof. If M' is a two-element matroid over R , let $s_{\leq n}(M)$ denote the alternating sum appearing in conditions (L2a,b). The matroid M has 6 minors with two elements. By adding

$$d_{\leq n}(M(\emptyset)) - d_{\leq n}(M(1)) - d_{\leq n}(M(2)) - d_{\leq n}(M(3))$$

to the three quantities in the proposition, we obtain the three values $s_{\leq n}(M \setminus a)$ for the deletions; by adding instead

$$d_{\leq n}(M(123)) - d_{\leq n}(M(12)) - d_{\leq n}(M(13)) - d_{\leq n}(M(23)),$$

we recover the three values $s_{\leq n}(M/a)$ for the contractions. So it is equivalent to prove that either of these sets of three attains its minimum multiple times.

We use induction on n . As base case we take $n = 0$, and have that $s_{\leq 0}(M') = 0$ for any M' . So let $n > 0$. Suppose first, as A varies over subsets of $[3]$, that $d_n(M(A))$ depends only on $|A|$. In this case, the three sums of form

$$d_n(M(\emptyset)) - d_n(M(a)) - d_n(M(b)) + d_n(M(ab))$$

are equal (as of course are the three sums of form

$$d_n(M(c)) - d_n(M(ac)) - d_n(M(bc)) + d_n(M(abc)).$$

Therefore the differences $s_{\leq n}(M \setminus a) - s_{\leq n-1}(M \setminus a)$ are all equal, and the induction step succeeds.

So suppose this is not the case, and there are two sets A and B with $|A| = |B| \in \{1, 2\}$, for which $d_n(M(A)) \neq d_n(M(B))$. We will proceed assuming that $|A| = |B| = 1$; the argument for the other case is exactly analogous (in fact, the two cases are exchanged by replacing M by its dual). Since there are only two possible values for $d_n(M(A))$ with $|A| = 1$, namely $d_n(M(\emptyset))$ and $d_n(M(\emptyset)) - 1$, two of the $d_n(M(A))$ with $|A| = 1$ are equal and are unequal to the third. Without loss of generality suppose $d_n(M(1)) = d_n(M(2)) \neq d_n(M(3))$. By condition (L2b), it follows that $s_{\leq n}(M \setminus 1) = s_{\leq n}(M \setminus 2) = 0$. Since $s_{\leq n}(M \setminus 3)$ is nonnegative by condition (L2a), this completes the induction for finite n .

Finally, the case $n = \infty$ holds because if any $d_{\leq \infty}(A)$ is finite, then $d_{\leq n}(A)$ must be eventually constant and equal to $d_{\leq \infty}(A)$. If the minimum in the proposition is not ∞ , there is nothing to prove; if this minimum is finite, the claim follows on replacing n by a sufficiently large finite number. \square

Suppose given a matroid M over R with ground set E . For $A \subseteq E$, define p_A to be $d_{\leq n}(M(A))$. Applying Proposition 5.5 to all three-element minors of M : the result can be restated to say that the *tropicalizations* of the relations

$$(5.3) \quad p_{Ab}p_{Acd} - p_{Ac}p_{Abd} + p_{Ad}p_{Abc} = 0$$

hold of the numbers p_{\bullet} , where we continue abbreviating $A \cup \{b, c\}$ as Abc and similarly.

For background on tropical geometry, see [15]. We say a bare minimum here: tropicalization is a procedure transforming algebraic varieties to tropical varieties, combinatorial “shadows” thereof, which are the sets of points on which the tropicalizations of all elements of their ideal of defining equations vanish. In our situation without a valued field, the tropicalization of a polynomial $f = \sum_{a \in A} c_a x^a$ in variables x_1, \dots, x_d is said to vanish at those points (x_i) where, of linear forms $\sum_i a_i x_i$ corresponding to the monomials in f , the minimum value is attained by two or more of the forms.

The relations (5.3) are among the Plücker relations for the full flag variety (of type A). A Plücker relation is a quadratic relation among the p_{\bullet} arising from the straightening algorithm for Young tableaux. The set of all Plücker relations forms a tropical basis for the full flag variety, and the set of relations (5.4) below with $|A| = |B| = r$ forms a tropical basis for the Grassmannian $\text{Gr}(r, n)$. We now describe them combinatorially, for which we need a preliminary notion of exchanges. Given subsets A and B of E with $|A| \leq |B|$, partition them into fixed and exchangable subsets $A = A_f \amalg A_e$, $B = B_f \amalg B_e$ so that $A \cap B$ is a subset of A_f and B_f , and $|A_e| + |B_e| = |B \setminus A| + 1$. Let $\text{exch}(A_f, A_e, B_f, B_e)$ be the set of pairs $(A'_f \cup A'_e, B'_f \cup B'_e)$ where A'_e and B'_e are a partition of $A_e \cap B_e$, and A'_e is disjoint from A_f and B'_e from B_f . Then

$$(5.4) \quad \sum_{(A', B') \in \text{exch}(A_f, A_e, B_f, B_e)} \pm p_{A'} p_{B'}$$

where the sign \pm takes its values so as to enforce antisymmetric behaviour of the indices. Note that one term of the sum (5.4) is $p_A p_B$.

Proposition 5.6. *Define $p_A = d_{\leq n}(M(A))$, where M is a matroid over R , and n is a natural or ∞ . Then the collection of p_A satisfies every tropical Plücker relation with $|A_e| = 1$.*

In particular, the point $(p_A : |A| = r)$ lies on the *Dressian* $\text{Dr}(r, n)$, which is one Grassmannian-like space in tropical geometry: it is the parameter space for tropical linear spaces [13]. That is, there is a tropical linear space determined by $(p_A : |A| = r)$. Equivalently, if the hypersimplex is given a regular subdivision wherein the height of vertex A is p_A , all maximal faces of this subdivision are matroid polytopes.

Corollary 5.7. *Let M be a matroid over a DVR (R, \mathfrak{m}) . Then the function $A \mapsto \dim_{R/\mathfrak{m}} M(A)$ makes the generic matroid of M into a valuated matroid, in the sense of Dress and Wenzel [9].*

To be precise, our sign convention is the opposite of the one adopted in [9]; for perfect agreement we would have to negate this function. But our sign convention is frequently adopted in tropical geometry, see e.g. [15].

Proof. Choose $n \gg 0$ sufficiently larger than the greatest length of any finite length summand of a module $M(A)$. The lengths of $M(A) \otimes R/\mathfrak{m}^n$ for A not a spanning set of the generic matroid are sufficiently greater than these lengths when A is a spanning set, since A is a spanning set of the generic matroid if and only if $M(A)$ has R as a summand.

The axiom of Dress and Wenzel for the valuation v of a valuated matroid is that, given bases A and B and $a \in A \setminus B$, there exists $b \in B \setminus A$ so that $A \setminus \{a\} \cup \{b\}$ and $B \cup \{a\} \setminus \{b\}$ are bases, and such that

$$v(A) + v(B) \geq v(A \setminus \{a\} \cup \{b\}) + v(B \cup \{a\} \setminus \{b\}).$$

The fact that the sets on the right hand side are bases follows from our choice of n , and the inequality is immediate from the minimum in the Plücker relation

$$\sum_{b \in (B \setminus A) \cup \{a\}} \pm p_{A \setminus \{a\} \cup \{b\}} p_{B \cup \{a\} \setminus \{b\}} = 0$$

being attained multiply, since $p_A p_B$ is one of its terms. \square

Conjecture 5.8. *The collection of p_A in Proposition 5.6 satisfies every tropical Plücker relation, i.e. gives a point on the tropical full flag variety.*

We expect that there is a proof of Conjecture 5.8 which, like our proof of Proposition 5.6, uses only the tropical equations shown in Proposition 5.5, and needs no further matroidal arguments. That is, we expect that any family p_A of elements of $\mathbb{R} \cup \{\infty\}$ satisfying the tropicalizations of all relations (5.3) defines a point of the tropical full flag variety (a variety that has been unfortunately little studied).

Remark 5.9. We expect that matroids over the ring of integers in the Puiseux series, $R = \bigcup_{n \geq 1} \mathbb{K}[[t^{1/n}]]$, should directly produce tropical objects with coordinates in \mathbb{Q} , when the length of $R/(t^a)$ is taken as a for $a \in \mathbb{Q}$; and that it is possible to use other valued rings similarly. Verifying this, and extending those parts of the theory which have relied on Noetherianity, is left for future work. \diamond

Proof of Proposition 5.6. In any Plücker relation with $|A_e| = 1$, the constraint $|A_e| + |B_e| = |B \setminus A| + 1$ implies that B_e is all of $B \setminus A$. So, once A and B are chosen, there is just one exchange relation for each one-element subset (i.e. element) of $A \setminus B$.

We will proceed by induction on $|A \setminus B| + |B \setminus A|$. If A is a subset of B , or if $|A \setminus B| = |B \setminus A| = 1$, there is no nontrivial Plücker relation. Thus the first nontrivial case is $|A \setminus B| = 1$ and $|B \setminus A| = 2$, and this is equation (5.3), which we have established as a base case.

The second nontrivial case is $|A \setminus B| = |B \setminus A| = 2$, and we again handle this case separately. In this case, let $F = A \cap B$, which equals B_f and is one element short of A_f . Suppose without loss of generality that $(A \setminus B) \cup (B \setminus A) = \{1, 2, 3, 4\}$. Then the tropicalized Plücker relation to be proved involves the three terms

$$p_{F12} + p_{F34}, \quad p_{F13} + p_{24}, \quad p_{F14} + p_{F23}.$$

Consider the six sums $p_{F \cup S_1} + p_{F \cup S_2} + p_{F \cup S_3}$, where S_1, S_2 , and S_3 are subsets of $\{1, 2, 3, 4\}$ of respective sizes 2, 2, and 1, whose union is $\{1, 2, 3, 4\}$, and such that 1 is the unique element appearing twice. There are six of these sums (not twelve, because the sum is the same even if S_1 and S_2 are exchanged). Among them there are three sums in which p_{F12} appears. The remaining two summands in these sums are the tropicalized terms of a Plücker relation (5.3), so their minimum is attained twice. The same goes for the sums in which p_{F13} appears, or p_{F14} .

From there, it follows that if the minimum value of all six of these sums was not attained by, say, $(S_1, S_2, S_3) = (13, 14, 2)$, it would be attained at both $(13, 24, 1)$ and $(23, 14, 1)$, and then by subtracting the common $p_{F \cup \{1\}}$ we would be finished. Accordingly, and by symmetry permuting $\{2, 3, 4\}$, we may assume that $(13, 14, 2)$, $(12, 14, 3)$, and $(12, 13, 4)$ all attain the minimum. In particular, they are all equal, and we rearrange to

$$p_{F12} - p_{F2} = p_{F13} - p_{F3} = p_{F14} - p_{F4}.$$

The same argument can be repeated with any of the elements of $\{2, 3, 4\}$ taking the place of 1. So if none of those gives the relation sought, we may conclude $p_{Fij} - p_{Fj} = p_{Fik} - p_{Fk}$ for every i, j, k in $\{1, 2, 3, 4\}$. But then

$$(p_{F12} - p_{F2}) + (p_{F34} - p_{F4}) = (p_{F14} - p_{F4}) + (p_{F23} - p_{F2})$$

so that $p_{F12} + p_{F34} = p_{F14} + p_{F23}$, and by symmetry $p_{F13}p_{F24}$ is equal to both of these as well. This finishes the case $|A \setminus B| = |B \setminus A| = 2$.

We finally proceed to the remaining cases, where $|B \setminus A| > 2$. For convenience, write $C = A_e \cup B_e$. It is reasonable to do this because, when $|A_e| = 1$, there is only one distinct Plücker relation for C : different partitions of it back into A_e and B_e of the correct sizes yield the same relation.

Let $c_1 \neq c_2$ be elements of C . Let D be a single-element subset of $A_f \setminus B$, if that set is nonempty, and let $D = \emptyset$ otherwise. Consider the set P of triples of sets $S = (S_1, S_2, S_3)$ where S_1, S_2 , and S_3 are subsets of C with $|S_1| = 1$, $|S_2| = |C| - 2$, $|S_3| = |C| - 1$, and such that the multiset union of S_1, S_2 and S_3 contains c_1 and c_2 with multiplicity 1, and each element of $C \setminus \{c_1, c_2\}$ with multiplicity 2. To each $S \in P$ associate the sum

$$\sigma(S) \doteq p_{A_f \cup S_1} + p_{B_f \cup D \cup S_2} + p_{B_f \cup S_3}.$$

A triple $S \in P$ is determined by two elements of C , namely the unique element a of S_1 and the unique element b of $C \setminus S_3$. We write $S_{a,b}$ for this triple, and observe that P contains exactly those $S_{a,b}$ with either $a = b$ or $a \notin \{c_1, c_2\}$, $b \in \{c_1, c_2\}$.

In particular, P contains three elements $S_{a,c}$ for each $a \in C \setminus \{c_1, c_2\}$. The sums $\sigma(S_{a,c})$ for these elements are the constant $p_{A_f \cup \{a\}}$ plus the tropicalizations of the three terms of an instance of (5.3) if D is empty, respectively the three terms of a Plücker relation where the sets corresponding to $A \setminus B$ and $B \setminus A$ each have size 2 if D is a singleton. Accordingly, the minimum value of $\sigma(S_{a,c})$ as c varies is attained twice.

Similarly, P contains $|C| - 1$ elements $S_{c,b}$ for each $b \in \{c_1, c_2\}$. The sums $\sigma(S_{c,b})$ for these elements are the constant $p_{B_f \cup C \setminus \{b\}}$ added to tropicalizations of the terms in another Plücker relation, where c_1 has been removed from whichever of A and B it was in, and $D \subseteq A$ has been added to B . This Plücker relation is one of those covered by the inductive hypothesis, since we've shrunk the symmetric difference of A and B . So the minimum value of $\sigma(S_{c,b})$ as c varies is also attained twice.

Once more, for the $|C|$ elements of P of the form $S_{c,c}$, each sum $\sigma(S_{c,c})$ is the constant $p_{B_f \cup C \setminus \{c_1, c_2\}}$ plus the tropicalization of a term in the Plücker relation whose tropical vanishing we are concerned with. So our objective is to show that the minimum value of $\sigma(S_{c,c})$ is attained twice.

Now let c_1 and c_2 be chosen so that the number of pairs $a \neq b$ for which $\sigma(S_{a,b})$ attains the minimum value $x = \min_{S \in P} \sigma(S)$ is as small as possible. We will prove that the minimum value of $\sigma(S_{c,c})$ is indeed attained twice. Suppose not. We then claim that there exists $a \notin \{c_1, c_2\}$ such that $\sigma(S_{a,c_1}) = \sigma(S_{a,c_2}) = x$. If this were false, choose $a \neq b$ so that $\sigma(S_{a,b}) = x$ (this must be possible, because if $\sigma(S_{a,b}) = x$ only when $a = b$ then the minimum of either $S_{a,c}$ or $S_{c,b}$ as c varies, whichever is appropriate, is attained just once.) By the structure of P , b must be c_1 or c_2 ; wlog let it be c_1 . That is, we are assuming $\sigma(S_{a,c_1}) = x$. Then by assumption $\sigma(S_{a,c_2}) > x$, so by the three-term Plücker relation, $\sigma(S_{a,a}) = x$. Moreover there must exist $a' \neq a$ such that $\sigma(S_{a',c_1}) = x$, by the other Plücker relation. If $a' = c_1$ then we have a contradiction with our first assumption (that the minimum is attained twice); otherwise we repeat for a' the argument we made for a and have a contradiction with our second.

Thus, we have $\sigma(S_{a,c_1}) = \sigma(S_{a,c_2}) = x$. Now, by assumption, at least one $c \in \{c_1, c_2\}$ has $\sigma(S_{c,c}) > x$; without loss of generality let it be c_1 . Let P' be defined like P except using $\{a, c_2\}$ where P uses $\{c_1, c_2\}$. For each $S = (S_1, S_2, S_3)$ in P such that $a \in S_2$, there is a corresponding $S' = (S_1, S_2 \setminus \{a\} \cup \{c_1\}, S_3)$ in P' , with

$$\sigma(S) - \sigma(S') = p_{B_f \cup D \cup S_2} - p_{B_f \cup D \cup S_2 \setminus \{a\} \cup \{c_1\}}.$$

Therefore, if S attains the minimum value x of $\sigma(S)$ over P , then S' attains the minimum value of $\sigma(S')$ over P' (unless this new minimum value is strictly less than $x - p_{B_f \cup D \cup S_2} + p_{B_f \cup D \cup S_2 \setminus \{a\} \cup \{c_1\}}$, in which case only one $S_{a,b} \in P'$ attains it, which contradicts our choice of c_1 and c_2). But in $\{S_{a,b} \in P : a \neq b\}$ we have two elements S_{a,c_1} and S_{a,c_2} without counterparts in P' , both of which attain the minimum, whereas in $\{S_{a,b} \in P' : a \neq b\}$ we have the counterpart of S_{c_1,c_1} , which does not attain the minimum. So this is also a contradiction to our choice of c_1 and c_2 , and our Plücker relation is proved in this case, completing the proof. \square

6. GLOBAL STRUCTURE OF MATROIDS OVER A DEDEKIND DOMAIN

Throughout this section R will be a Dedekind domain. Understanding the local ring case, we can now give a necessary and sufficient condition for which pairs of modules can occur in axiom (M1).

Proposition 6.1. *Let N and N' be f.g. R -modules. There exists a surjection $N \rightarrow N'$ with cyclic kernel if and only if there exists such a surjection $N_{\mathfrak{m}} \rightarrow N'_{\mathfrak{m}}$ after localizing at each maximal prime \mathfrak{m} of R , and*

- if $\text{rk}(N) - \text{rk}(N') = 0$ then $[N_{\text{proj}}] = [N'_{\text{proj}}]$ in $\text{Pic}(R)$, whereas
- if $\text{rk}(N) - \text{rk}(N') = 1$ then $[N] = [N']$ in $\text{Pic}(R)$.

To test whether surjections exist in the localizations, we have the criterion in Proposition 5.2.

Proof. Necessity. Localization is a base change so preserves axiom (M1). If $\text{rk}(N) = \text{rk}(N')$, then the kernel of $N \rightarrow N'$ is contained in N_{tors} , so that $N_{\text{proj}} \cong N'_{\text{proj}}$, and so their classes are equal. If $\text{rk}(N) = \text{rk}(N') + 1$ then the kernel of $N \rightarrow N'$ must be a cyclic rank 1 R -module, which up to isomorphism is R . Therefore $[N] = [R] + [N'] = [N']$ by the definition of $K_0(R)$.

Sufficiency. Note first that $\text{rk}(N) - \text{rk}(N') \in \{0, 1\}$, because the same is true in every localization. Let us suppose that $\text{rk}(N) = \text{rk}(N')$. Then $[N_{\text{proj}}] = [N'_{\text{proj}}]$ implies $N_{\text{proj}} \cong N'_{\text{proj}}$, by Proposition 3.3. Moreover, N_{tors} and N'_{tors} are the direct sums of their localizations. The kernel of each of the given maps $N_{\mathfrak{m}} \rightarrow N'_{\mathfrak{m}}$ is contained in the torsion $(N_{\mathfrak{m}})_{\text{tors}} = (N_{\text{tors}})_{\mathfrak{m}}$, so a map $(N_{\text{tors}})_{\mathfrak{m}} \rightarrow (N'_{\text{tors}})_{\mathfrak{m}}$ is induced. The direct sum of all these maps is a map $N_{\text{tors}} \rightarrow N'_{\text{tors}}$ which is still a surjection; its kernel is a sum of cyclic modules with disjoint support, which is still cyclic. Taking the direct sum with an isomorphism $N_{\text{proj}} \rightarrow N'_{\text{proj}}$ yields the requisite map $N \rightarrow N'$.

Now suppose that $\text{rk}(N) = \text{rk}(N') + 1$. In this case, construct a set function $M : \mathcal{B}(1) \rightarrow R\text{-Mod}$ so that $M(\emptyset) = N'_{\text{tors}}$ and $M(1) = N_{\text{tors}}$. Note that $M^*(\emptyset)$ and $M^*(1)$ both have rank 0 and therefore trivial projective part. Moreover, there exist localized surjections with cyclic kernel $M(\emptyset)_{\mathfrak{m}} \rightarrow M(1)_{\mathfrak{m}}$ for each \mathfrak{m} . This is by Proposition 4.6(c) (or because localization is flat), because $M(\emptyset)_{\mathfrak{m}}$ and $M(1)_{\mathfrak{m}}$ are the modules of a one-element matroid over $R_{\mathfrak{m}}$, the dual of the matroid built from $N_{\mathfrak{m}} \rightarrow N'_{\mathfrak{m}}$.

Using the previous case, there exists a surjection with cyclic kernel $M(\emptyset) \rightarrow M(1)$. That is, M is a matroid over R . By Proposition 4.6(a), dualizing this matroid and taking the direct sum with the empty matroid for N'_{proj} yields a one-matroid over R whose objects are N and N' . There thus exists a surjection with cyclic kernel $N \rightarrow N'$. \square

For a complete description of the structure of matroids over R we must treat axiom (M2) as well. It turns out there are no (ring-theoretically) global conditions on such squares, and thus on matroids over R , further to those imposed by axiom (M1).

Proposition 6.2. *Let $M(\emptyset)$, $M(1)$, $M(2)$, and $M(12)$ be f.g. R -modules. There exist four surjections with cyclic kernels forming a pushout square*

$$\begin{array}{ccc} M(\emptyset) & \longrightarrow & M(1) \\ \downarrow & \lrcorner & \downarrow \\ M(2) & \longrightarrow & M(12) \end{array}$$

if and only if the same is true after localizing at each maximal prime \mathfrak{m} , and the source and target of each map satisfy Proposition 6.1.

Proof. Necessity. Trivial in view of Proposition 6.1 and the fact that pushout squares localize to pushout squares.

Sufficiency. Fix a pushout square for each localization; label its maps as follows.

$$\begin{array}{ccc} M(\emptyset)_{\mathfrak{m}} & \xrightarrow{f_{\mathfrak{m}}} & M(1)_{\mathfrak{m}} \\ g_{\mathfrak{m}} \downarrow & \lrcorner & \downarrow g'_{\mathfrak{m}} \\ M(2)_{\mathfrak{m}} & \xrightarrow{f'_{\mathfrak{m}}} & M(12)_{\mathfrak{m}} \end{array}$$

It is enough to construct two maps $M(\emptyset) \rightarrow M(1)$ and $M(\emptyset) \rightarrow M(2)$ which localize correctly everywhere, for then we may choose $M(12)$ to be their pushout, since pushouts localize to pushouts.

Suppose first that one of $M(1)$ and $M(2)$ has the same rank as $M(\emptyset)$, without loss of generality that $M(2)$ does. By Proposition 6.1, we may construct a map $\phi : M(\emptyset) \rightarrow M(1)$ so that there exist isomorphisms $i_{\mathfrak{m}}(A)$ for each prime \mathfrak{m} and $A = \emptyset, 1$ making the squares

$$\begin{array}{ccc} M(\emptyset)_{\mathfrak{m}} & \xrightarrow{\phi_{\mathfrak{m}}} & M(1)_{\mathfrak{m}} \\ i_{\mathfrak{m}}(\emptyset) \downarrow & & \downarrow i_{\mathfrak{m}}(1) \\ M(\emptyset)_{\mathfrak{m}} & \xrightarrow{f_{\mathfrak{m}}} & M(1)_{\mathfrak{m}} \end{array}$$

commute. Now, by the proof of Proposition 6.1, we construct $M(0) \rightarrow M(2)$ as the direct sum of the restriction of the given $g_{\mathfrak{m}}$ to the torsion submodule of $M(\emptyset)_{\mathfrak{m}}$, and the identity map $M(0)_{\text{proj}} \rightarrow M(0)_{\text{proj}}$. It changes nothing to precompose each of these restrictions of $g_{\mathfrak{m}}$ by the corresponding $i_{\mathfrak{m}}(\emptyset)$. Doing this yields a commutative square

$$\begin{array}{ccc} M(\emptyset)_{\mathfrak{m}} & \xrightarrow{\psi_{\mathfrak{m}}} & M(2)_{\mathfrak{m}} \\ i_{\mathfrak{m}}(\emptyset) \downarrow & & \downarrow \\ M(\emptyset)_{\mathfrak{m}} & \xrightarrow{g_{\mathfrak{m}}} & M(2)_{\mathfrak{m}} \end{array}$$

and pasting this square to the last one shows that we have constructed the two maps $M(\emptyset) \rightarrow M(1)$ and $M(\emptyset) \rightarrow M(2)$ which localize as desired.

The remaining case is the one in which the ranks of $M(1)$ and $M(2)$ are both less than that of $M(\emptyset)$. In this case, like the second case of Proposition 6.1, we will proceed via dualization, and then via a similar argument. In brief, we may first

construct a map $M^*(2) \rightarrow M^*(12)$ which localizes correctly, up to intertwining with some isomorphisms. Then since the map $M^*(1) \rightarrow M^*(12)$ doesn't involve a rank drop, we may construct it as a direct sum of localizations on the torsion parts using the same isomorphisms. This gives us a diagram $M^*(1) \rightarrow M^*(12) \leftarrow M^*(2)$ which localizes correctly at every maximal prime. Finally, we may temporarily insert any suitable module in place of $M^*(\emptyset)$, for instance the pullback, and then dualize the resulting matroid over R . Discarding the ersatz $M(12)$ gives us maps $M(2) \leftarrow M(\emptyset) \rightarrow M(1)$ which localize correctly, as desired. \square

6.1. Quasi-arithmetic matroids. If M is a matroid over \mathbb{Z} , then we can define a corank function of M as the corank function of the generic matroid $M \otimes_{\mathbb{Z}} \mathbb{Q}$ described above, that is

$$\text{cork}(A) = \dim(M(A) \otimes_{\mathbb{Z}} \mathbb{Q}) - \dim(M(E) \otimes_{\mathbb{Z}} \mathbb{Q}).$$

As before, we let $M(A)_{\text{tors}}$ denote the torsion submodule (subgroup, in this case) of $M(A)$. Then we define

$$m(A) \doteq |M(A)_{\text{tors}}|.$$

Corollary 6.3. *The triple (E, cork, m) is a quasi-arithmetic matroid, i.e. (E, cork) defines a matroid, and m satisfies the following properties:*

- (A1) *Let be $A \subseteq E$ and $b \in E$; if b is dependent on A , then $m(A \cup \{b\})$ divides $m(A)$; otherwise $m(A)$ divides $m(A \cup \{b\})$;*
- (A2b) *if $A \subseteq B \subseteq E$ and B is a disjoint union $B = A \cup F \cup T$ such that for all $A \subseteq C \subseteq B$ we have $\text{rk}(C) = \text{rk}(A) + |C \cap F|$, then*

$$m(A) \cdot m(B) = m(A \cup F) \cdot m(A \cup T).$$

Furthermore it satisfies the following property:

- (A2a) *if $A, B \subseteq E$ and $\text{rk}(A \cup B) + \text{rk}(A \cap B) = \text{rk}(A) + \text{rk}(B)$, then $m(A) \cdot m(B)$ divides $m(A \cup B) \cdot m(A \cap B)$*

Proof. Since $\text{Pic}(\mathbb{Z})$ is trivial, this is immediate from Propositions 5.2, 5.4, 6.1 and 6.2. \square

This corollary establishes that matroids over \mathbb{Z} recover many of the essential features of the second author's theory of *arithmetic matroids* from [6]. To be precise, the objects we have recaptured are *quasi-arithmetic matroids*: see Remark 6.4. In fact the two objects are not truly equivalent, in that the information contained in matroids over \mathbb{Z} is richer.

Remark 6.4. An arithmetic matroid is an object satisfying all the axioms of an arithmetic matroid plus a further one, namely the positivity property (P) of [3]. The axioms of quasi-algebraic matroids are arithmetic ones, pertaining to integer divisibility, whereas (P) has geometric motivation. To be precise, (P) is included as an axiom in order to force positivity of the arithmetic Tutte polynomial $M_A(x, y)$. Its geometric nature is that the numbers whose positivity it demands are, in the realizable case, numbers of components of certain strata in the corresponding toric arrangement. Its coefficients also have two natural but nontrivial combinatorial interpretations [6, 3].

The additional property (A2a) appeared in an earlier choice of the axioms [6, first arXiv version]. \diamond

7. THE TUTTE-GROTHENDIECK GROUP

In this section we continue to let R be a Dedekind domain. All matroids over R in this section are full-rank. The word “matroid” will mean “matroid over R ” from here through the end of the proof of Lemma 7.7, except when we speak of a generic matroid.

As we defined the operations of deletion and contraction in Section 2, any element may be deleted or contracted. However, suppose $a \in E$ is a (*generic*) *coloop* of a matroid M over R , that is a coloop of the generic matroid, equivalently an element such that $M(E \setminus \{a\})$ has a nontrivial projective summand. In this case, $M \setminus a$ is not full-rank. The dual of this situation is the case where a is a (*generic*) *loop*, i.e. a loop of the generic matroid, and one contracts a .

Essentially following Brylawski [4], define the *Tutte-Grothendieck group* of matroids over R , which we here denote $K(R\text{-Mat})$, to be the abelian group generated by a symbol \mathbf{T}_M for each isomorphism class M of full-rank matroids over R with nonempty ground set, modulo the relations

$$\mathbf{T}_M = \mathbf{T}_{M \setminus a} + \mathbf{T}_{M/a}$$

whenever a is not a generic loop or coloop (so that we avoid the above situations).

We will use exponential notation for the basis elements of the monoid ring of an additively-written monoid, even when the monoid contains torsion. Thus for instance we may write the group ring of $\text{Pic}(R)$ as $\mathbb{Z}[v^{\mathcal{E}} : \mathcal{E} \in \text{Pic}(R)]$; in particular, it is to be understood that the objects \mathcal{E} add identically whether they are exponents or no, and thus we do not mean the polynomial ring in $|\text{Pic}(R)|$ variables.

Consider the “Grothendieck” monoid generated by the classes of all f.g. R -modules with relations given by direct sums

$$[N] + [N'] = [N \oplus N']$$

(and not relations corresponding to all short exact sequences, as is usually done). Propositions 3.3 and 3.4 allow us to describe the structure of this monoid. It has one generator for each indecomposable torsion module R/\mathfrak{m}^k and one for each rank 1 projective. There are no relations involving the former generators, and the latter have the relations $[P] + [Q] = [R] + [P \otimes Q]$, so that the submonoid they generate is isomorphic to a submonoid of the group $\mathbb{Z} \oplus \text{Pic}(R) \cong K_0(R)$. The monoid ring of this Grothendieck monoid can be presented as

$$\mathbb{Z}[v^{\mathcal{E}}, X, z_{\mathfrak{m},k} : \mathcal{E} \in \text{Pic}(R), \mathfrak{m} \in \text{MaxSpec}(R), k \in \mathbb{Z}_{>0}].$$

The class of a module N in the monoid ring becomes $v^{[N]} X^{\text{rank}(N)} z^N$, where for a torsion module $N \cong R/\mathfrak{m}_1^{k_1} \oplus \cdots \oplus R/\mathfrak{m}_n^{k_n}$, we define z^N to be $z_{\mathfrak{m}_1 k_1} \cdots z_{\mathfrak{m}_n k_n}$.

Theorem 7.1 says that the Tutte-Grothendieck group $K(R\text{-Mat})$ of matroids over R , which gains a ring structure from direct sum, is not dissimilar to the tensor square of this monoid ring, where the two tensor factors account for modules in a matroid and its dual. Recall in this connection that $\text{cork}_M(A) = \text{nullity}_{M^*}(E \setminus A)$ and $M(A)_{\text{tors}} = M^*(E \setminus A)_{\text{tors}}$.

Theorem 7.1. *The Tutte-Grothendieck group $K(R\text{-Mat})$ is a ring without unity, with product given by $\mathbf{T}_M \cdot \mathbf{T}_{M'} = \mathbf{T}_{M \oplus M'}$. As a ring it injects into*

$$S = \mathbb{Z}[v^{\mathcal{E}}, w^{\mathcal{E}}, x, y, z_{\mathfrak{m},k} : \mathcal{E} \in \text{Pic}(R); \mathfrak{m} \in \text{MaxSpec}(R), k \in \mathbb{Z}_{>0}].$$

That is, S is obtained from the tensor square of the group ring of the Picard group (in the v and w variables) by freely adjoining two variables (x and y) plus one more variable (namely $z_{\mathfrak{m},k}$) for each indecomposable torsion module R/\mathfrak{m}^k .

Under the above injection, the class of M is

$$(7.1) \quad \mathbf{T}_M = \sum_{A \subseteq E} v^{[M(A)]} w^{[M^*(E \setminus A)]} (x-1)^{\text{cork}_M(A)} (y-1)^{\text{nullity}_M(A)} z^{M(A)_{\text{tors}}}.$$

We next compare this theorem with the case of matroids over a field, where the Tutte-Grothendieck invariant is the familiar Tutte polynomial \mathbf{T}_M . (We will make more such comparisons in Section 7.1.) Theorem 7.1 in fact recovers the Tutte polynomial if our $\text{MaxSpec}(R)$ is instead read as the set of prime ideals of height 1, which is the same thing for a DVR. Over a field there are no primes of height 1 and so the z variables, as well as the v and w variables, fall away. This concordance with the standard Tutte polynomial is what motivated our choice to use powers of $(x-1)$ and $(y-1)$ in \mathbf{T}_M , rather than x and y . We will continue in this fashion below, regarding as “monomials” in S those polynomials which are products of powers of the v , w , and z variables and of $x-1$ and $y-1$.

The terms of the invariant \mathbf{T}_M has a global consistency property following directly from Proposition 4.7.

Corollary 7.2. *Consider the ring homomorphism $\text{cl} : S \rightarrow \mathbb{Z}[u^{\mathcal{E}} : \mathcal{E} \in \text{Pic}(R)]$ given by $\text{cl}(v^{\mathcal{E}}) = \text{cl}(w^{\mathcal{E}}) = u^{\mathcal{E}}$, $\text{cl}(x-1) = \text{cl}(y-1) = 1$, and $\text{cl}(z^N) = u^{-[N]}$. Then $\text{cl}(\mathbf{T}_M)$ is a scalar multiple of $u^{\text{cl}(M)}$.*

Proof. Going term by term in equation (7.1), this is immediate from the equality

$$\text{cl}(M) = [M(A)] + [M^*(E \setminus A)] - [M(A)_{\text{tors}}]. \quad \square$$

We have excluded empty matroids from the definition of $K(R\text{-Mat})$ because there are no linear relations relating them to matroids with nonempty ground set: the unique element in a matroid on one element must be a loop or coloop. Thus, constructing the Tutte-Grothendieck group in the presence of zero-element matroids would duplicate the subgroup of $K(R\text{-Mat}) \subseteq S$ lying within $\mathbb{Z}[u^{\mathcal{E}}, v^{\mathcal{E}}, z_{\mathfrak{m},k}]$. However, if classes \mathbf{T}_M for empty matroids are defined via equation (7.1), these classes behave correctly under the multiplication defined in Theorem 7.1.

This feature can be ascribed to our choice to consider only the Tutte-Grothendieck group, rather than the Tutte-Grothendieck ring where one also imposes the relations $\mathbf{T}_M \cdot \mathbf{T}_{M'} = \mathbf{T}_{M \oplus M'}$ from the outset. The group of the Tutte-Grothendieck ring of an arbitrary bidecomposition, to use the terminology of Brylawski [4], is in general a quotient of the Tutte-Grothendieck group. But in this case the discrepancy is small: the ring associated to all matroids over R is equal to S , the group associated to matroids over R with at least one element.

Since decomposing a matroid M over a ring into $M \setminus i$ and M/i is not a unique decomposition in the sense of [4], and the irreducibles for direct sum are not all single-element matroids, Theorem 7.1 does not follow directly from the bidecomposition methods of [4].

Proof of Theorem 7.1. Let S be the ring defined in the theorem. To keep distinct the objects which we have not yet proven isomorphic, let $[M]$ represent the class of M in $K(R\text{-Mat})$, reserving \mathbf{T}_M for the element of S defined in (7.1).

Consider the map $\mathbf{T} : K(R\text{-Mat}) \rightarrow S$ given by $\mathbf{T}([M]) = \mathbf{T}_M$. We have that \mathbf{T} is a homomorphism of rings, because the deletion-contraction relations and multiplicativity relations hold among the various \mathbf{T}_M . Both of these are straightforward to check, and correspond to easy operations on equation (7.1). The deletion-contraction relation on an element a is proved by splitting the sum into one sum containing the terms with $a \notin A$ and another containing the terms with $a \in A$. Multiplicativity under direct sum is proved by expanding the product of equation (7.1) for M and M' , and collecting into a single sum over $A \amalg A' \subseteq E \amalg E'$.

With that, we come to the involved part of the proof, which is to show \mathbf{T} an injection. Our approach will be to construct a family \mathcal{I} of matroids M whose polynomials \mathbf{T}_M are linearly independent in S , and use deletion-contraction relations to expand every matroid in terms of \mathcal{I} . This will allow every linear relation among the \mathbf{T}_M to be lifted to a relation among the $[M]$ via expansion in terms of \mathcal{I} , proving injectivity. We will moreover be able to conclude that the image of \mathbf{T} is the span of the images of the matroids in \mathcal{I} .

As we use the deletion-contraction relations, we will make frequent use of induction on the size of the ground set. In fact, our main technique will be to embed a matroid M as a minor of another, M' , and then relate M to another minor of M' of the same size plus a collection of smaller minors. But if the ground set has size 1, this will not be as useful: the unique element of a 1-element matroid is necessarily either a loop or a coloop, hence we cannot get construct a deletion-contraction relation involving a smaller matroid as a minor. This will be our base case, and require a different argument. We have broken out the arguments expanding these matroids in terms of \mathcal{I} into two lemmas, Lemma 7.5 and Lemma 7.7.

The following construction is relevant to both cases. Linearly extend the divisibility relation on the ideals of R to a total order \leq such that for ideals I, J, K , $I \leq J$ implies $IK \leq JK$. For each class $\mathcal{E} \in \text{Pic}(R)$, let $N_{\mathcal{E}}$ equal R/I , where I is the \leq -least ideal of R whose class in $\text{Pic}(R)$ is \mathcal{E}^{-1} . This produces a fixed cyclic torsion module $N_{\mathcal{E}}$ representing each class $\mathcal{E} \in \text{Pic}(R)$. Note that every submodule N' of $N_{\mathcal{E}}$ is also the representative of its own class, $N' = N_{[N']}$. Define the single-element matroid $L_{\mathcal{E}}$ by $L_{\mathcal{E}}(\emptyset) = N_{\mathcal{E}}$ and $L_{\mathcal{E}}(1) = 0$. The dual matroid $L_{\mathcal{E}}^*$ therefore has $L_{\mathcal{E}}^*(\emptyset) = R$ and $L_{\mathcal{E}}^*(1) = N_{\mathcal{E}}$. (L and L^* can be taken to stand for “loop” and “coloop”.) Also, let \emptyset_N be the empty matroid associated to a torsion R -module N .

For a torsion module N , define a second sort of loop K_N by taking $K_N(\emptyset) = N$ and $K_N(1)$ to be the quotient of N by its largest invariant factor.

We construct the set \mathcal{I} as follows.

$$\begin{aligned}
 \mathcal{I} = & \{K_N : N \text{ is torsion}\} \\
 & \cup \{\emptyset_N \oplus L_{\mathcal{E}} \oplus L_0^{\oplus a} : N \text{ is torsion}, \mathcal{E} \in \text{Pic}(R), a \geq 0\} \\
 (7.2) \quad & \cup \{\emptyset_N \oplus L_{\mathcal{F}}^* \oplus (L_0^*)^{\oplus b} : N \text{ is torsion}, \mathcal{F} \in \text{Pic}(R), b \geq 0\} \\
 & \cup \{\emptyset_N \oplus L_{\mathcal{E}} \oplus L_0^{\oplus a} \oplus L_{\mathcal{F}}^* \oplus (L_0^*)^{\oplus b} : N \text{ is torsion}, \mathcal{E}, \mathcal{F} \in \text{Pic}(R), a, b \geq 0\}
 \end{aligned}$$

To analyze linear relations in \mathcal{I} , we give the ring S a monomial order wherein $(x-1)$ and $(y-1)$ are greater than any monomial not containing them, and moreover $(x-1) > (y-1)$. Then if M has a unique basis, as all the matroids in \mathcal{I} do, the initial term of \mathbf{T}_M is the term contributed to the sum in (7.1) by the complement

of the unique basis of M . For the matroid

$$M = \emptyset_N \oplus L_{\mathcal{E}} \oplus L_0^{\oplus a} \oplus L_{\mathcal{F}}^* \oplus (L_0^*)^{\oplus b}$$

the complement of the unique basis is sent to $N \oplus P \oplus R^b$, where P is the rank 1 projective module of class $\mathcal{F} \in \text{Pic}(R)$. For the dual of this matroid, the analogous module is $N \oplus Q \oplus R^a$ where Q is the rank 1 projective of class \mathcal{E} . Therefore the initial term of \mathbf{T}_M is $v^{\mathcal{F}} w^{\mathcal{E}} (x-1)^{b+1} (y-1)^{a+1} z^N$. Similarly, if instead of M we had taken one of the matroids from the two previous lines in \mathcal{I} , the $v^{\mathcal{F}} (x-1)^{b+1}$ or $w^{\mathcal{E}} (y-1)^{a+1}$ terms would be omitted. All the monomials in these three classes are distinct.

Finally, the initial term of \mathbf{T}_{K_N} contains $(y-1)$ to the power 1 and $(x-1)$ to the power 0. It follows that any nontrivial \mathbb{Z} -linear relation among the classes of elements of \mathcal{I} may contain only these matroids and others of smaller leading terms: that is, it may involve only one-element matroids whose unique element is a loop.

Temporarily let \mathcal{I}_1 be the set of the matroids K_N , and \mathcal{I}_2 the set of matroids of form $\emptyset_N \oplus L_{\mathcal{E}}$, so that together every matroid in \mathcal{I} whose unique element is a loop is in \mathcal{I}_1 or \mathcal{I}_2 . Suppose there was a nontrivial linear dependence among the classes of these matroids. (The sets \mathcal{I}_1 and \mathcal{I}_2 share some elements, but since we have taken their union as *sets* this is not a problem.) The class of each matroid in $\mathcal{I}_1 \cup \mathcal{I}_2$ is of the form $v^{\mathcal{E}} w^{\mathcal{F}} (z^N + (x-1)z^{N'})$ where N and N' are torsion R -modules; moreover, there is only one element of \mathcal{I}_1 and one of \mathcal{I}_2 with given N . Therefore, if there is any linear relation, there must be a minimal one of the form

$$(7.3) \quad \sum_{j=1}^k [M_{1,j}] - \sum_{j=1}^k [M_{2,j}] = 0$$

where $M_{i,j} \in \mathcal{I}_i$, all $M_{i,j}(\emptyset)$ have the same class in $\text{Pic}(R)$, and $M_{1,j}(\emptyset) = M_{2,j}(\emptyset)$. The equality also implies that the product of the annihilators in R of the kernels of $M_{1,j}(\emptyset) \rightarrow M_{1,j}(1)$ equals the corresponding product of annihilators for the kernels of $M_{2,j}(\emptyset) \rightarrow M_{2,j}(1)$. All of these annihilators have the same class in $\text{Pic}(R)$. The latter product is I^k , where I is the annihilator of $N_{\mathcal{E}}$. Therefore, at least one of the ideals in the former product must be less than or equal to I in \leq order. But I is the \leq -minimal ideal of its class, and so these ideals must all equal I , so that all coefficients on the left side of (7.3) are zero and the relation is trivial. Thus \mathcal{I} is dependent, as claimed. \square

Remark 7.3. Examining the span of the classes in \mathcal{I} allows us to describe the image of $K(R\text{-Mat})$ within S . If zero-element matroids are included, then $K(R\text{-Mat})$ is the subring generated by the elements $(x-1)v^{\mathcal{E}}$ and $(y-1)w^{\mathcal{E}}$ for each $\mathcal{E} \in \text{Pic}(R)$ together with $v^{[N]}w^{[N]}z^N$ for each torsion module N . If zero-element matroids are excluded, it is the kernel within that subring of the map to \mathbb{Z} sending $v^{\mathcal{E}} \mapsto 1$, $w^{\mathcal{E}} \mapsto 1$, $(x-1) \mapsto -1$, $(y-1) \mapsto -1$, $z_{\mathbf{m},k} \mapsto 1$ (since in any non-empty matroid, there are equally many sets whose symmetric difference with a basis is of even and of odd size). \diamond

The following subsidiary lemma will afford us useful flexibility for manipulating single-element matroids in proving Lemma 7.5.

Lemma 7.4. *Let \mathcal{E} be a class in $\text{Pic}(R)$, and P a finite set of maximal primes of R . There exists a cyclic torsion R -module N whose support is disjoint from P such that $[N] = \mathcal{E}$.*

Proof. This is a restatement of a standard lemma on ideal factorizations, see e.g. [5, Corollary 4.9]. In the notation of that corollary, we let \mathfrak{a} be any representative of the class \mathcal{E} and let \mathfrak{b} be the product of the members of P , and the module we seek is $N = R/\mathfrak{c}$. \square

Lemma 7.5. *If M is a one-element matroid satisfying any one of the following, the class $[M] \in K(R\text{-Mat})$ lies in the span of the classes of matroids in the set \mathcal{I} of (7.2).*

- (a) $M(\emptyset) = P \oplus N$ and $M(1) = N \oplus C$, where P is rank 1 projective, N is torsion and C is cyclic, and the supports of N and C are disjoint.
- (b) $M(\emptyset) = N \oplus C$ and $M(1) = N$, where N is torsion and C is cyclic, and the supports of N and C are disjoint.
- (c) $M(\emptyset) = P \oplus N'$, $M(1) = N$, where P is rank 1 projective, N is torsion, and N' is the quotient of N by its largest invariant factor.
- (d) either M or M^* sends \emptyset to $P \oplus T \oplus N'$ and $\{1\}$ to $T \oplus N$, where P is rank 1 projective, N is torsion, N' is the quotient of N by its largest invariant factor, and the support of T is disjoint from that of N .
- (e) either M or M^* sends \emptyset to $N \oplus T$ and $\{1\}$ to T , where N is cyclic and T is torsion.
- (f) M is any one-element matroid.

Proof. A matroid M' on two elements whose generic matroid is the uniform matroid $U_{1,2}$ gives rise to a linear relation among its four one-element minors,

$$(7.4) \quad [M' \setminus 1] + [M'/1] = [M'] = [M' \setminus 2] + [M'/2].$$

We will use this to prove the cases of the lemma sequentially, reducing each to a linear combination of matroids in \mathcal{I} and matroids in previous cases. For visibility we will specify these M' by drawing the commutative square

$$\begin{array}{ccc} M'(\emptyset) & \longrightarrow & M'(1) \\ \downarrow & & \downarrow \\ M'(2) & \longrightarrow & M'(12) \end{array}$$

In each case M' can be checked to be a matroid by Proposition 6.2. The non-local conditions reduce to checking that $[M'(1)] = [M'(2)]$ in $\text{Pic}(R)$.

To (a). Let N' be the quotient of N by its largest invariant factor. First, suppose that the support of C is disjoint from the supports of N and $L_{[C]}(1)$. In that case, the following square specifies a matroid M' . The modules $N \oplus C$ and $N' \oplus L_C(1)$ have the same class in $\text{Pic}(R)$ by construction. And because of the assumption on supports, modulo each maximal ideal \mathfrak{m} , either the top map has kernel $R_{\mathfrak{m}}$ and the right one is trivial, or the same is true of the left and bottom maps respectively. This ensures that the localizations of the square are pushouts.

$$\begin{array}{ccc} P \oplus N & \longrightarrow & N \oplus L_{[C]}(1) \\ \downarrow & & \downarrow \\ N \oplus C & \longrightarrow & N' \end{array}$$

The left minor $M' \setminus 1$ is the matroid we are interested in; the bottom minor $M'/2$ and the right minor $M'/1$ are both among the matroids K_N in \mathcal{I} ; and the top minor $M' \setminus 2$ is among the $\emptyset_N \oplus L_{\mathcal{F}}^*$. So the relation (7.4) proves the result in this case.

Next, if we lack the support assumptions on C , we are assured the existence of a cyclic module C' of support disjoint from $L_{[C]}(1)$ and N , of the same class in $\text{Pic}(R)$ as C , by Lemma 7.4. In this case, we repeat the argument with the following square, which can similarly be checked to give a matroid M'' .

$$\begin{array}{ccc} P \oplus N & \longrightarrow & N \oplus C' \\ \downarrow & & \downarrow \\ N \oplus C & \longrightarrow & N' \end{array}$$

Now $M'' \setminus 1$ is the matroid of interest, the minors $M'/2$ and $M'/1$ are among the matroids K_N , and $M' \setminus 2$ is in the span of \mathcal{I} by the last paragraph. Therefore, using (7.4) again, we have proved case (a).

To (b). Let R/I be the largest invariant factor of N . Let J be a nonzero ideal contained in I chosen so that $[J] = -[C]$ in $\text{Pic}(R)$, and the supports of R/J and $L_{[C]}(1)$ are disjoint; this exists by Lemma 7.4. Then R/J is the largest invariant factor of $N \oplus R/J$. Having done this, both of the following squares give matroids, where P is a suitably chosen rank 1 projective module.

$$\begin{array}{ccc} M' : & P \oplus N & \longrightarrow N \oplus L_{[C]}(1) \\ & \downarrow & \downarrow \\ & N \oplus R/J & \longrightarrow N \end{array} \quad \begin{array}{ccc} M'' : & P \oplus N & \longrightarrow N \oplus C \\ & \downarrow & \downarrow \\ & N \oplus R/J & \longrightarrow N \end{array}$$

Subtracting the relation (7.4) for the two matroids, we express the class of $M''/1$, which is the matroid of interest, as a linear combination of the classes of $M'' \setminus 2$, $M'/1$, and $M' \setminus 2$. But, of these, $M'' \setminus 2$ is one of the matroids appearing in part (a), $M'/1$ is of form $\emptyset_N \oplus L_{\mathcal{E}}$, and $M' \setminus 2$ is of form $\emptyset_N \oplus L_{\mathcal{F}}^*$. This proves case (b).

To (c). Use Lemma 7.4 to produce a cyclic module C whose class in $\text{Pic}(R)$ is $[N] - [N']$ and whose support is disjoint from that of N . Then the following square gives a matroid M' .

$$\begin{array}{ccc} P \oplus N' & \longrightarrow & N' \oplus C \\ \downarrow & & \downarrow \\ N & \longrightarrow & N' \end{array}$$

Here, $M'/2$ is of form $\emptyset_N \oplus L_{\mathcal{E}}$, and $M'/1$ is covered by case (b) of the lemma, and $M' \setminus 2$ is covered by case (a). So (7.4) proves case (c).

At this point, we pause to take note that the matroids of form $\emptyset_N \oplus L_{\mathcal{E}}$ and $\emptyset_N \oplus L_{\mathcal{E}}^*$ and the matroids encompassed by this last case (c) are the duals of the K_N . These are all the matroids in \mathcal{I} of one element. Moreover, in equation (7.4), dualizing M' dualizes the four minors. Therefore, for the rest of this proof, arguing that a class $[M]$ is in the linear span of classes of matroids in \mathcal{I} will imply the same for the class $[M^*]$ of the dual.

To (d). As stated just above, it is sufficient to treat the case where M is as described, not its dual. For this we use induction on the number of invariant factors of the

torsion module T . If it has none, it is the zero module and we are in case (a). Otherwise, let T' be the quotient of T by its largest invariant factor.

Let N'' be the quotient of N' by its largest invariant factor. Lemma 7.4 gives cyclic modules C and D whose classes in $\text{Pic}(R)$ take the necessary values, and whose supports are disjoint from the supports of other appearing modules as necessary, in order for the following squares to specify matroids M' and M'' .

$$\begin{array}{ccc} M' : P \oplus N' \oplus T & \longrightarrow & N' \oplus T \oplus C \\ \downarrow & & \downarrow \\ N \oplus T & \longrightarrow & N' \oplus T' \end{array} \quad \begin{array}{ccc} M'' : P \oplus N'' \oplus T' & \longrightarrow & N' \oplus T \oplus C \\ \downarrow & & \downarrow \\ N' \oplus T' \oplus D & \longrightarrow & N' \oplus T' \end{array}$$

Here $M'/1$ equals $M''/1$, and these can be cancelled out of the two corresponding invocations of (7.4), leaving a linear relation among the six other minors. Of these, $M' \setminus 1$ is the matroid of interest. $M'' \setminus 1$ is the matroid to which we will apply the induction hypothesis: when applying it we take the new module T to be T' , which has one invariant factor fewer than (the old) T , and the new modules $N' \rightarrow N$ to be $N'' \rightarrow N' \oplus D$. The remaining minors are dealt with: $M'/2$ is among the K_N , $M''/2$ is dealt with in case (b) of this lemma, $M' \setminus 2$ in case (a), and $M'' \setminus 2$ in case (c). Therefore the induction goes through and we have proved case (d).

To (e). Again we may assume M (not M^*) is as described. We use induction on the maximum k such that, for some maximal prime \mathfrak{m} contained in the support of N , there are k cyclic summands of $T_{\mathfrak{m}}$ longer than $N_{\mathfrak{m}}$. If $k = 0$, then N is the largest invariant factor of the part of $N \oplus T$ with the same support, and M falls under case (d).

Otherwise, let F be the direct sum of the localizations of the largest invariant factor of T of length exceeding the corresponding localization of N , and let T' be T/F . Lemma 7.4 provides cyclic modules C and D so that the following squares are matroids. (For the top maps of the squares, this is where the fact that $\dim_{R/\mathfrak{m}} F_{\mathfrak{m}} \geq \dim_{R/\mathfrak{m}} N_{\mathfrak{m}}$ is used.)

$$\begin{array}{ccc} M' : P \oplus N & \longrightarrow & T \oplus C \\ \downarrow & & \downarrow \\ N \oplus \mathbf{T} & \longrightarrow & T \end{array} \quad \begin{array}{ccc} M'' : P \oplus N & \longrightarrow & T \oplus C \\ \downarrow & & \downarrow \\ N \oplus \mathbf{T}' \oplus D & \longrightarrow & T' \end{array}$$

Again, $M'/2$ equals $M''/2$, and two invocations of (7.4) give a linear relation among the remaining minors. Of these $M' \setminus 2$ is the matroid of interest, $M'' \setminus 2$ is covered by the inductive hypothesis, and all of $M'/1$, $M''/1$, $M' \setminus 1$, and $M'' \setminus 1$ fall under case (d). This proves case (e).

To (f). Either M or M^* is of global rank 0, and we may assume it is M . Here we use one further induction. Let k be the maximum, over maximal primes \mathfrak{m} , of the number of times 01 or 10 appear as substrings of the sequence $d_{\bullet}(\phi_{\mathfrak{m}})$ associated in Section 5 to the map $\phi_{\mathfrak{m}}$ in the localized matroid $M \otimes R_{\mathfrak{m}}$. As a base case, if $k \leq 1$, then each $\phi_{\mathfrak{m}}$, and therefore ϕ , is a quotient by a cyclic summand; this is case (e).

Otherwise, let N be the quotient of $M(\emptyset)$ by its largest invariant factor. With C provided by Lemma 7.4 as usual, we have a matroid M' given by

$$\begin{array}{ccc} P \oplus N & \longrightarrow & M(1) \oplus C \\ \downarrow & & \downarrow \\ M(\emptyset) & \longrightarrow & M(1) \end{array}$$

The minor $M'/2$ is M . The minor $M' \setminus 2$ is covered by the induction hypothesis: if ψ is the map in this matroid, then for each \mathbf{m} the sequence $d_\bullet(\psi_{\mathbf{m}})$ is obtained from $d_\bullet(\phi_{\mathbf{m}})$ by replacing the final infinite run of 0s by 1s, so one of the substrings 10 is lost. The minors $M'/1$ and $M' \setminus 1$ both are handled by case (e). This proves case (f) and finishes our discussion of one-element matroids. \square

We approach the reduction of matroids on several elements to our basis in Lemma 7.7 below, in several steps as we did in Lemma 7.5. The bulk of our discussion here will pertain to matroids with one generic basis; in the terminology of [6], these are called *molecules* (since for matroids over a field all molecules are direct sums of *atoms*, i.e. one-element matroids). First we state two subsidiary technical lemmas.

Lemma 7.6. *If M is a molecule on ground set E and a a generic coloop in it, then $M(A)_{\text{proj}} = M(E \setminus a)_{\text{proj}} \oplus M(Aa)_{\text{proj}}$ for every $A \subseteq E \setminus \{a\}$.*

Proof. This is clear for $A = E \setminus \{a\}$, so by induction on the size of the complement of A we need only establish the statement A given the statement for Ab . The rank drop between $M(A)$ and $M(Ab)$ equals that between $M(Aa)$ and $M(Aab)$. If this rank drop is zero, then we are done because $M(Ab)_{\text{proj}} = M(A)_{\text{proj}}$ and $M(Aab)_{\text{proj}} = M(Aa)_{\text{proj}}$. If the rank drop is one, then given maps making a pushout square

$$\begin{array}{ccc} M(A) & \xrightarrow{\phi} & M(Ab) \\ \downarrow & & \downarrow \\ M(Aa) & \xrightarrow{\phi'} & M(Aab) \end{array}$$

the kernels of ϕ and ϕ' are isomorphic projective modules; call one of them P . Then, in $K_0(R)$, we have

$$[M(A)] = [M(Ab)] \oplus [\ker \phi] = [P] \oplus [M(Aab)] \oplus [\ker \phi'] = [P] \oplus [M(Aa)]$$

and the K -class of a projective module determines it. \square

Lemma 7.7. *If M is a matroid on ground set E satisfying any of the following, then the class $[M] \in K(R\text{-Mat})$ lies in the span of the classes of the matroids in \mathcal{I} .*

- (a) M is a direct sum of one-element matroids.
- (b) The only generic basis of M is \emptyset .
- (c) M is a molecule.
- (d) M is arbitrary.

Proof. As in the last proof, we will manipulate our matroid M by fabricating larger matroids in which M appears as a minor. But we have more room to maneuver, as matroids on fewer elements than M may also appear in the deletion-contraction relations, and induction on the ground set size shows that the classes of these are in the span of \mathcal{I} .

To (a). For this step of the argument we will use another induction, on the number of one-element summands of a matroid M which are not L_0 or L_0^* , possibly excluding *one* summand of each of the forms $L_{\mathcal{E}}$ and $L_{\mathcal{F}}^*$. (There may also be a summand which is an empty matroid for a torsion module; this will be inert and have no effect on our argument). In the base case, M is a direct sum of some copies of L_0 , possibly a single $L_{\mathcal{E}}$, some copies of L_0^* , possibly a single $L_{\mathcal{E}}^*$, and some empty matroid \emptyset_N ; this is an element of \mathcal{I} .

As inductive step, we will use deletion-contraction relations to increase the number of such summands in two ways, one of which applies to any direct sum of at least two one-element matroids. One of our constructions will replace a direct sum of two one-element summands of the same generic rank by a matroid of form $L_{\mathcal{E}} \oplus L_0$ or $L_{\mathcal{F}}^* \oplus L_0^*$. The other will replace a direct sum of two one-element summands of unequal generic ranks with some $L_{\mathcal{E}} \oplus L_{\mathcal{F}}^*$.

For the former construction suppose we have a two-element molecule N , without loss of generality having two coloops, which is a summand of M ; write $M = N \oplus K$. For convenience suppose the ground set of N is $\{1, 2\}$.

The basis of N is \emptyset , and $N(\emptyset)$ has the form $P \oplus R \oplus T$, where P is a rank 1 projective module and T is some torsion module. Fix maps $\phi : N(\emptyset) \rightarrow N(1)$, $\psi : N(\emptyset) \rightarrow N(2)$. By making the non-free analogue of a change of basis in this splitting $P \oplus R$ if necessary, we can suppose that neither of the saturations of $\ker \phi$ nor $\ker \psi$ is contained in R or P . Now embed N in a realizable matroid N' on $\{1, 2, 3, 4\}$ so that $N = N' \setminus \{3, 4\}$, the map $N'(\emptyset) \rightarrow N(3)$ is the quotient map $P \oplus R \oplus T \rightarrow L_{[P]}(1) \oplus R \oplus T$ on the first factor, the map $N'(\emptyset) \rightarrow N(4)$ is the quotient $P \oplus R \oplus T \rightarrow P \oplus T$ on the second, and the rest of N' is completed by taking pushouts.

By construction, none of the kernels of the maps with source $N'(\emptyset)$ in this realization has its saturation contained in another such saturation, so that the quotient of $N'(\emptyset)$ by the sum of two such kernels has rank 0. Thus, the generic matroid of N' is $U_{2,4}$. Thus, the direct sum $M' = N' \oplus K$ is $U_{2,4}$ plus a molecule. We will use deletion-contraction relations to break M' down in two ways, the knowledge of the generic matroid of M' assuring us that we are not choosing loops or coloops. On one hand, use (in sequence) the elements 3 of M' , 4 of $M' \setminus 3$, 1 of $M' \setminus 3/4$, 1 of $M'/3$, and 2 of $M' \setminus 1/3$. On the other, use the elements 1 of M' , 2 of $M' \setminus 1$, 3 of $M' \setminus 1/2$, 3 of $M'/1$, and 4 of $M' \setminus 3/1$. This gives us equalities of classes in the Grothendieck group:

$$\begin{aligned} & [M' \setminus 3, 4] + [M' \setminus 1, 3/4] + [M' \setminus 3/1, 4] + [M' \setminus 1, 2/3] + [M' \setminus 1/2, 3] + [M'/1, 3] \\ &= [M'] \\ &= [M' \setminus 1, 2] + [M' \setminus 1, 3/2] + [M' \setminus 1/2, 3] + [M' \setminus 3, 4/1] + [M' \setminus 3/1, 4] + [M'/1, 3] \end{aligned}$$

The term $[M'/1, 3]$ cancels, and all of the remaining terms aside from $[M' \setminus 3, 4]$ and $[M' \setminus 1, 2]$ are classes of matroids on fewer elements, so they are in the span of the classes of \mathcal{I} by our top-level induction. The matroid $M' \setminus 3, 4$ is our original M .

Finally, $M' \setminus 1, 2$ has more summands than M which are L_0 or L_0^* : there is a new such summand in $M' \setminus 1, 2$ on the element 4. So it is covered by one of our inductions as well.

Turning to the latter construction, we will in fact need to invoke a second induction, on the rank of the generic matroid of M , that is the size of its generic basis. We set this up decreasingly, so the base case is when M has only coloops: in this case, M has no loop and this construction cannot in fact imply.

Continuing, we suppose M has a two-element summand N , say on ground set $\{1, 2\}$, which is itself the sum of a matroid N_1 on its coloop 1, and a matroid N_2 on its loop 2. Again we write $M = N \oplus K$.

By choosing any maps and computing the pushout, we may construct a matroid \tilde{N}_2 on ground set $\{2, 4\}$ where $\tilde{N}_2(\emptyset) = N_2(\emptyset)$, $\tilde{N}_2(2) = N_2(2)$, and $\tilde{N}_2(4) = N_2(\emptyset)_{\text{tors}} \oplus L_{[P]}(1)$ where $P = (N_2(\emptyset))_{\text{proj}}$. Its generic matroid will be $U_{1,2}$. With the dual of this construction we also construct a matroid \tilde{N}_1 on ground set $\{1, 3\}$ with generic matroid $U_{1,2}$, where $\tilde{N}_1(3) = N_1(\emptyset)$, $\tilde{N}_1(13) = N_1(1)$, and $\tilde{N}_1(1) = N_1(1) \oplus L_{\mathcal{E}}(1)$ where $\mathcal{E} = [N_1(\emptyset)] - [N_1(1)]$.

We will construct N' as a perturbation of $\tilde{N} \doteq \tilde{N}_1 \oplus \tilde{N}_2$, as follows. Fix realizations of \tilde{N}_1 and \tilde{N}_2 , so that the induced realization of \tilde{N} provides four maps ϕ_1, \dots, ϕ_4 with cyclic kernel from the module $\tilde{N}(\emptyset)$, corresponding respectively to the atoms $1, \dots, 4$ covering \emptyset in $\mathcal{B}(4)$. The kernels of ϕ_1 and ϕ_3 are both contained in $\tilde{N}_1(\emptyset)$, while the kernels of ϕ_2 and ϕ_4 are contained in $\tilde{N}_2(\emptyset)$; all of them are isomorphic to R as R -modules. The module $\tilde{N}_1(\emptyset)$ is the direct sum of a projective rank 1 summand P , and a torsion module. There exists an injection $\psi : P \hookrightarrow \ker \phi_2 \cap \ker \phi_4$. This can be composed with the embedding $\ker \phi_2 \cap \ker \phi_4 \subseteq \tilde{N}(\emptyset)$ and summed with zero maps on the other summands to produce a map $\psi : \tilde{N}(\emptyset) \rightarrow \tilde{N}(\emptyset)$. The map $(\text{id} + \psi) : \tilde{N}(\emptyset) \rightarrow \tilde{N}(\emptyset)$ is then “upper triangular” and hence an automorphism. Let x be a generator of $\ker \phi_3$, and define a new map ϕ'_3 from $\tilde{N}(\emptyset)$ to be the quotient by the submodule $\langle x + \psi(x) \rangle$. Finally, let N' be the matroid on ground set $\{1, 2, 3, 4\}$ with $N'(\emptyset) = \tilde{N}(\emptyset)$ and whose maps and other modules induced as pushouts of ϕ_1, ϕ_2, ϕ'_3 , and ϕ_4 .

Our perturbation of ϕ_3 to ϕ'_3 has arranged that $\text{cork}_{N'}(13) = 0$. On the other hand, is $3 \notin A$ then $N'(A)$ is unchanged from $\tilde{N}(A)$; if A contains 3 and one of 2 or 4 but not 1 then $N'(A) \cong \tilde{N}(A)$ by construction of ψ ; and $N'(3) \cong \tilde{N}(3)$ as well, since $\text{id} + \psi$ is an automorphism. In particular the generic matroid of N is the rank 2 matroid on $\{1, 2, 3, 4\}$ with no loops whose only nontrivial parallelism class is $\{2, 4\}$.

Let $M' = N' \oplus K$. We have deletion-contraction relations giving the following equalities:

$$\begin{aligned} & [M' \setminus 3, 4] + [M' \setminus 3/4] + [M' \setminus 1, 2/3] + [M' \setminus 1/2, 3] + [M'/1, 3] \\ &= [M'] \\ &= [M' \setminus 1, 2] + [M' \setminus 1/2] + [M' \setminus 3, 4/1] + [M' \setminus 3/1, 4] + [M'/1, 3] \end{aligned}$$

The term $[M'/1, 3]$ cancels, and the two terms before it in each line are matroids on fewer elements. The matroid $M' \setminus 3/4$ is our original M , since M was the same minor of $\tilde{N} \oplus K$ and we haven't altered the relevant modules in it. The matroid $N' \setminus 1/2$, for the analogous reason, is the direct sum of $L_{[P]}^*$ on the element 4, $L_{\mathcal{E}}$

on the element 3, and an empty matroid, so $M' \setminus 1/2$ improves on the quantity counted in the induction we introduced at the start of this case (a). The remaining matroids, $M' \setminus 3, 4$ and $M' \setminus 1, 2$, are also direct sums of one-element matroids, and they both have generic rank 2, so they are covered by our latest-introduced induction. Altogether, this finishes case (a).

To (b). We will use induction on the number of elements of E which are not the ground set of a one-element direct summand. The base case is part (a).

We construct a matroid M' on $E \amalg \{\eta\}$ which will agree in most of its modules with the direct sum of M and a loop $\emptyset \mapsto 0$, $\{\eta\} \mapsto 0$. In particular M'/η will be M . We let $M'(\emptyset)$ be obtained from $M(\emptyset)$ by replacing its largest invariant factor with a projective module of the same class in $\text{Pic}(R)$. For each $b \in E$, we use Lemma 7.4 to produce a cyclic module $C(b)$ of disjoint support from any module in M and so that $[C(b)] + [M(b)] = [M(\emptyset)]$ in $\text{Pic}(R)$, and then set $M'(b) = M(b) \oplus C(b)$. In any other case set $M'(A) = M(A \setminus \eta)$, where η is not necessarily in A .

Our choices of $M'(\emptyset)$ and the modules $M'(b)$ for singletons are exactly as is needed so that all the pairs $M'(\emptyset)$, $M'(b)$ satisfy the K -theoretic condition of Proposition 6.1. For the other covering relations of subsets of $E \amalg \{\eta\}$, both modules are rank 0 so the K -theoretic condition is trivially satisfied. The localization conditions are essentially inherited from M . Since the summands $C(b)$ have support disjoint from any of the other modules under consideration, they don't interfere in this respect. The alteration we have made to $M'(\emptyset)$ replaces a final infinite string of 0s by 1s in the sequences d_\bullet associated to the maps $M'(\emptyset)_m \rightarrow M'(b)_m$; the resulting sequence is still of the sort allowed by Proposition 5.2. These same facts about the localizations also suffice to establish Proposition 6.2, in which only the local considerations of Proposition 5.4 are relevant.

The generic matroid of M' is $U_{1,|E|+1}$. Therefore, no deletion of M' with more than one element is a molecule, and we may freely use the deletion-contraction relation on such deletions. Let a be any element of E . Splitting M into three minors by deletion-contraction in two ways, we have

$$[M' \setminus \eta \setminus a] + [M' \setminus \eta/a] + [M'/\eta] = [M'] = [M' \setminus a/\eta] + [M' \setminus a/\eta] + [M'/a]$$

so that, cancelling the common deletion,

$$[M' \setminus \eta/a] + [M'/\eta] = [M' \setminus a/\eta] + [M'/a].$$

Here, the minor M'/η is our matroid M of interest. The matroid M'/a has a one-element summand with ground set $\{\eta\}$ together with whichever one-element summands a had, so it is subsumed by our induction hypothesis. The other two matroids are on fewer elements. This proves case (b).

To (c). Here we use induction on the number of generic coloops and on the size of the number of elements which don't generate single-element direct summands.

Suppose that a is a generic coloop of M . Then $M(E \setminus a)$ has a rank 1 projective summand, call it P . By Lemma 2.8, the empty matroid \emptyset_P for P splits as a direct summand of $M \setminus a$. Name the other direct summand N .

Let C be a cyclic module which is sufficiently large that every cyclic summand of a module appearing in M is isomorphic to a quotient of C , and such that $[P] = [C]$ in $\text{Pic}(R)$. Let M' be a system of R -modules so that $M' \setminus \eta = M$; $M'/\eta/a = N \oplus \emptyset_C$; and $M'/\eta/a = M/a$. That is, M' is obtained from the direct sum \bar{M} of M and the one-element matroid $\emptyset \mapsto 0$, $\{\eta\} \mapsto 0$ by replacing a summand P by C at every set

containing η but not a . We will show that M' is a matroid using Propositions 6.1 and 6.2.

For Proposition 6.2, since \tilde{M} is a matroid, we need only check that the replacements of P by C don't interfere with the condition to be checked in Proposition 5.4. If \mathfrak{m} is a maximal prime, then the sequences $d_\bullet(\tilde{M}(A))$ and $d_\bullet(M'(A))$ are of course identical if no replacement has taken place, and if one has, they differ only in that $d_i(M'(A)) = d_i(\tilde{M}(A)) - 1$ for all $i \geq k$, where k is such that every sequence $d_\bullet(M(B))$ is constant from the k th position on. Replacing P by C can't cause any difference $d_i(M'(A)) - d_i(M'(Ab))$ to leave the range $\{0, 1\}$: if this difference were to be 2 then b must be η , and if it were to be -1 then b must be a , but neither of these situations occur in the construction. The replacement also doesn't change the quantity on the left side of the displays in (L2a) and (L2b) for any two-element minor M'' of M' , and hence doesn't undermine the truth of these conditions, unless the ground set of M'' is $\{a, \eta\}$, in which case that quantity is incremented. But in this event, by construction, the equality of (L2b) is attained in the corresponding minor of \tilde{M} for $d_{\leq k}$, and so (L2b) is still true of M'' .

For Proposition 6.1, all that remains to check are the equalities of classes in $\text{Pic}(R)$. There are two cases to consider which are not inherited from M or $N \oplus \emptyset_C$. One involves $M'(A)$ and $M'(A\eta)$ for $\eta \notin A$ and $a \notin A$, where the rank drop is 1, and $M'(A) = P \oplus N(A)$ and $M'(A\eta) = C \oplus N(A)$ have the same class in $\text{Pic}(R)$ by choice of C . The other involves $M'(A)$ and $M'(Aa)$ for $\eta \in A$ and $a \notin A$, where the rank drop is 0. In this case Lemma 7.6 gives that $M(A)_{\text{proj}} = P \oplus M(Aa)_{\text{proj}}$. Then $M'(A)_{\text{proj}} = (C \oplus M(Aa))_{\text{proj}} = M(Aa)_{\text{proj}}$ and $M'(Aa)_{\text{proj}} = M(Aa)_{\text{proj}}$ agree.

Thus M' is a matroid. In its generic matroid, all elements are loops or coloops except for η and a which generate a uniform matroid $U_{1,2}$. so we have deletion-contraction relations

$$[M' \setminus \eta] + [M'/\eta] = [M'] = [M' \setminus a] + [M'/a].$$

In this relation $M' \setminus \eta$ is our M . The matroid M'/η has a one-element direct summand on ground set $\{a\}$, so is encompassed by our second induction; the matroids $M' \setminus a$ and M'/a have a greater number of coloops than M , so are encompassed by our first. This proves case (c).

To (d). Repeatedly using deletion-contraction to break up any matroid with at least two bases, on any element which is not a loop or coloop, expresses the class of any matroid as a sum of classes of molecules. \square

7.1. Arithmetic Tutte polynomial and quasi-polynomial. When $R = \mathbb{Z}$, the Picard group is trivial. Then no information is lost in eliminating from the universal invariant \mathbf{T}_M the variables v, w by evaluating them at 1. In this way we get:

$$\overline{\mathbf{T}}_M = \sum_{A \subseteq E} (x-1)^{\text{cork}_M(A)} (y-1)^{\text{nullity}_M(A)} z^{M(A)_{\text{tors}}}.$$

Since every maximal ideal \mathfrak{m} is generated by a prime number p , we write $z_{p,k}$ for $z_{\mathfrak{m},k}$. By evaluating every variable $z_{p,k}$ at p^k , $\overline{\mathbf{T}}_M$ specializes to the arithmetic Tutte polynomial $\mathbf{M}_{\hat{M}}(x, y)$ of the quasi-arithmetic matroid \hat{M} defined by M :

$$\mathbf{M}_{\hat{M}}(x, y) = \sum_{A \subseteq E} m(A) (x-1)^{\text{rk}(E) - \text{rk}(A)} (y-1)^{|A| - \text{rk}(A)}.$$

This polynomial proved to have several applications to toric arrangements, partition functions, zonotopes, and graphs with labeled edges (see [16], [6]). Notice that an ordinary matroid \tilde{M} can be trivially made into an arithmetic matroid \hat{M} by setting all the multiplicities to be equal to 1, and then $\mathbf{M}_{\hat{M}}(x, y)$ is nothing but the classical Tutte polynomial $\mathbf{T}_{\tilde{M}}(x, y)$.

The polynomial $\mathbf{M}_{\hat{M}}(x, y)$ is not the universal deletion-contraction invariant of \hat{M} : for instance, the ordinary Tutte polynomial $\mathbf{T}_{\tilde{M}}(x, y)$ of the matroid \tilde{M} obtained from \hat{M} by forgetting of its arithmetic data is also a deletion-contraction invariant of \hat{M} , which is not determined by $\mathbf{M}_{\hat{M}}(x, y)$. This led the authors of [3] to define a *Tutte quasi-polynomial* $\mathbf{Q}_M(x, y)$, interpolating between $\mathbf{T}_{\tilde{M}}(x, y)$ and $\mathbf{M}_{\hat{M}}(x, y)$. This invariant is stronger, but still not universal, and more importantly, it is not an invariant of the arithmetic matroid, as it depends on the groups $M(A)_{\text{tors}}$ and not just on their cardinalities. We will now show that $\mathbf{Q}_M(x, y)$ is actually an invariant of the matroid over \mathbb{Z} , and write explicitly how to compute it from the universal invariant.

For every positive integer q , let us define a function V_q as $V_q(z_{p,k}) = 1$ if p^k divides q , while $V_q(z_{p,k}) = p^{k-j}$ if p^k does not divide q and $j \geq 0$ is the largest integer such that p^j divide q . We will denote by $V_q(z^{M(A)_{\text{tors}}})$ the product of the values assumed on the single variables. Then we define

$$\begin{aligned} \mathbf{Q}_M(x, y) &\doteq \sum_{A \subseteq E} (x-1)^{\text{cork}_{M(A)}} (y-1)^{\text{nullity}_{M(A)}} V_{(x-1)(y-1)}(z^{M(A)_{\text{tors}}}) = \\ &= \sum_{A \subseteq E} \frac{|M(A)_{\text{tors}}|}{|(x-1)(y-1)M(A)_{\text{tors}}|} (x-1)^{\text{rk}(E) - \text{rk}(A)} (y-1)^{|A| - \text{rk}(A)}. \end{aligned}$$

Since $(q + |G|)G = qG$ holds for any finite group G , it follows that $\mathbf{Q}_M(x, y)$ is a quasi-polynomial in $q = (x-1)(y-1)$. In particular, when $|M(A)_{\text{tors}}|$ divides $(x-1)(y-1)$, then the group $(x-1)(y-1)M(A)_{\text{tors}}$ is trivial and $\mathbf{Q}_M(x, y)$ coincides with $\mathbf{M}_{\hat{M}}(x, y)$; while when $|M(A)_{\text{tors}}|$ is coprime with $(x-1)(y-1)$, then $\mathbf{Q}_M(x, y)$ coincides with $\mathbf{T}_{\tilde{M}}(x, y)$. Then in some sense $\mathbf{Q}_M(x, y)$ interpolates between the two polynomials.

Notice that while $\mathbf{M}_{\hat{M}}$ and $\mathbf{T}_{\tilde{M}}(x, y)$ only depend just on the induced quasi-arithmetic matroid \hat{M} , $\overline{\mathbf{T}_M}$ and $\mathbf{Q}_M(x, y)$ are indeed invariants of the matroid over \mathbb{Z} , M . Also the *chromatic quasi-polynomial* and the *flow quasi-polynomial* defined in [3] are actually invariants of the matroid over \mathbb{Z} : by [3, Theorem 9.1] they are specializations of $\mathbf{Q}_M(x, y)$, and hence of the universal invariant \mathbf{T}_M .

REFERENCES

- [1] LAURA ANDERSON, EMANUELE DELUCCHI, *Foundations for a theory of complex matroids*, arXiv:1005.3560.
- [2] R. G. BLAND, M. LAS VERGNAS, *Orientability of matroids*, J. Combin. Theory Ser. B **24** (1978), 94–123.
- [3] PETTER BRÄNDÉN, LUCA MOCI, *The multivariate arithmetic Tutte polynomial*, arXiv:1207.3629 [math.CO].
- [4] THOMAS H. BRYLAWSKI, *The Tutte-Grothendieck ring*, Algebra Universalis **2** no. 1 (1972), 375–388, DOI: 10.1007/BF02945050.
- [5] KEITH CONRAD, *Ideal factorization*, www.math.uconn.edu/~kconrad/blurbs/gradnumthy/idealfactor.pdf

- [6] MICHELE D'ADDERIO AND LUCA MOCI, *Arithmetic matroids, Tutte polynomial, and toric arrangements*, arXiv:1105.3220 [math.CO], to appear on Advances in Mathematics.
- [7] M. D'ADDERIO, L. MOCI, *Graph colorings, flows and arithmetic Tutte polynomial*, arXiv:1108.5537.
- [8] C. DE CONCINI, C. PROCESI, *Topics in hyperplane arrangements, polytopes and box-splines*, Universitext, Springer-Verlag, New-York (2010), XXII+381 pp.
- [9] A. DRESS, W. WENZEL, *Valuated matroids*, Adv. Math. **93** (1992), no. 2, 214–250.
- [10] DAVID EISENBUD, *Commutative algebra with a view towards algebraic geometry*, Graduate Texts in Mathematics **150**, Springer, 1995.
- [11] WILLIAM FULTON, *Intersection theory*, second edition, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge*, Springer-Verlag, 1998.
- [12] MICHIEL HAZEWINKEL, *Handbook of Algebra, Volume 1*, Elsevier, 1995.
- [13] SVEN HERRMANN, MICHAEL JOSWIG, DAVID SPEYER, *Dressians, Tropical Grassmannians, and Their Rays*, arXiv:1112.1278, Forum Mathematicum (2012), DOI: 10.1515/forum-2012-0030.
- [14] IAN G. MACDONALD, *Symmetric functions and Hall polynomials*, 2nd ed., Oxford Mathematical Monographs, Oxford University Press, New York, 1995.
- [15] DIANE MACLAGAN, BERND STURMFELS, *Introduction to tropical geometry*, draft, homepages.warwick.ac.uk/staff/D.Maclagan/papers/TropicalBook.pdf
- [16] LUCA MOCI, *A Tutte polynomial for toric arrangements*, arXiv:0911.4823v4 [math.CO], to appear in Trans. Am. Math. Soc.
- [17] J. G. OXLEY, *Matroid Theory*, Oxford University Press, Oxford, 1992.
- [18] A. SOKAL, *The multivariate Tutte polynomial (alias Potts model) for graphs and matroids*, Surveys in combinatorics 2005, 173–226, London Math. Soc. Lecture Note Ser., 327, Cambridge Univ. Press, Cambridge, 2005.
- [19] W. T. TUTTE, *A contribution to the theory of chromatic polynomials*, Canadian J. Math., **6**: 80–91, 1954.
- [20] CHARLES WEIBEL, *The K-book: an introduction to algebraic K-theory*, <http://math.rutgers.edu/~weibel/Kbook.html>.